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Toward A General Model of Financial Markets*

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Abstract

This paper aims to discuss the possibilities of capturing efficient market hypothesis and behavioral finance under a general framework using the literature of decision theories and information sciences. The focus is centered on the broad definition of rationality, the imprecision and reliability of information. The main thesis advanced is that the root of behavioral anomalies comes from the imprecision and reliability of information. Modeling on basis of imprecision and reliability of information within the broad definition of rationality will lead us to a more general model of financial markets.

JEL Classification: G02, G10, G14, D81.
Keywords: Efficient markets; Behavioral finance; Decision theory; Information uncertainty.

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1 The Argument

The key argument of this paper takes its root from "fact" vs. "opinion". In an environment rife with heterogeneous opinions, behavioral biases naturally arise when agents deal with imprecise and partially true information. For example, suppose there are two urns, A and B, and two groups of people, G_1 and G_2. One of the urns contains 100 of \$1 bills and the other one 70 of \$1 bills. Everyone in G_1 will pick the urn A given that $P(A) > P(B)$ where $P(A)$ and $P(B)$ show the probabilities of total \$100 being in urns A and B. Now suppose, similar to the balls-in-box problem of Zadeh (2005), people in G_2 are given an opportunity to look at each box without being able to count the total amount in the boxes. Clearly, in this situation people in G_2 will have different opinions based on their visual perceptions. This simple illustration shows that, information becomes open to interpretation when it is not a precise fact as in the first case. In other words, heterogeneity is created by simply providing imperfect information.

In view of this simple mind experiment, we redefine rationality and suggest a more general framework of financial markets. Specifically, we use Zadeh (2011) classification of information, "numerical", "interval-valued", "second-order uncertain", "fuzzy" and "Z" information, based on its generality. We argue such that individuals are subjectively rational if they apply correct decision technique to each class of information separately rather than defining rationality based on only one decision technique such as the standard Savage's axioms of subjective expected utility for all classes of information. Also, the majority of existing financial market models do not take the reliability of information into account which is indeed a major determinant in real life decision process. Therefore, most of the existing financial models become too simplistic to account for real world with partially reliable, imprecise information. Once we increase the imprecision and at the same time decrease the reliability of information the environment becomes too complex to be explained by standard techniques.

With the same "fact" vs. "opinion" argument, we further argue that efficient market hypothesis and behavioral finance become special cases of this more general framework with the imprecision and reliability of information connecting them. More precisely, imprecise and partially reliable information triggers interaction between psychological factors to play a primary role in financial decision-making process and in generating “anomalies” while precise facts approximately lead to efficient market hypothesis. In order to forestall needless arguments, let us also mention that we do not claim that imprecision and reliability are the only factors connecting EMH and behavioral finance, but the most important factors.

1.1 Arguments of ‘Efficient Market Hypothesis’

The random walk movement of stock prices is first comprehensively formalized in Osborne (1959) by making a number of assumptions. One of the underlying assumptions of Osborne’s world is ‘logical decision’, in which investors are assumed to form expectations in a probabilistic manner and choose the course of action with a higher expected value. That is to say, investors form objective probabilities and make rational decisions as if they know each individual outcome. Another crucial assumption made by Osborne (1959) is an independence of decisions in the sequence of transactions of a single stock which leads to independent, identically distributed successive price changes. This implies that changes in prices can only come from unexpected new information. In this setting, central limit theorem assures that daily, weekly and monthly price changes converge to Gaussian or normal distribution which is later generalized by Mandelbrot (1963) to account for the empirical evidence of leptokurtic distributions of price changes. Stable Paretian distribution hypothesis is later supported by Fama (1965). Fama also argues that an independence assumption may still hold due to an existence of sophisticated traders even though the processes generating noise and new information are dependent. That is, an independence assumption is consistent with ‘efficient markets’ where prices at every point in time represent the best estimates of intrinsic values. The combination of independence and stable Paretian distribution allows him to argue that the actual prices adjust instantaneously to the changes in intrinsic value due to the discontinuous nature of stable Paretian distribution. Therefore, this version of efficient market hypothesis includes random walk theory as a special case. However, the first formal general economic argument of 'efficient markets' is given by Samuelson (1965) by focusing on the martingale property. Similar to the 'logical decision' assumption of Osborne (1959), Samuelson (1965) also assumes that people in financial markets make full use of the past probability distribution. Overall, the proponents of EMH essentially argued the use of a probability calculus as a foundation of economic analysis. This also justified an application of probability based decision techniques (e.g., expected utility theory) to financial modeling.

1.2 Arguments of 'Behavioral Finance'

The main arguments of behavioral finance stand in sharp contradiction to the logical decisions assumption of efficient market hypothesis in forming expectations. The early works of Kahneman and Tversky is a foundational block of this area of finance. In a series of experiments, they show that people use heuristic to decide under uncertainty and conjecture that the same heuristic plays an important role in the evaluation of uncertainty in real life. In their seminal paper, Kahneman and Tversky (1979) present a critique of expected utility theory and develop prospect theory as an alternative. At the same time, Shiller (1979) shows that long-term interest rates are too volatile to be justified by rational models. Inspired by Tversky and Kahneman's works, Thaler (1980) argues that consumers do not follow economic theory and proposes an alternative descriptive theory on the basis of prospect theory. Similarly, Shiller (1981) argues that stock prices fluctuate too much to be justified by subsequent dividend changes. All of these arguments and findings sharply contradicted efficient market hypothesis and shaped the emergence of a new field. Finally, Bondt and Thaler (1985) marked the birth of behavioral finance with empirical evidence of overreaction hypothesis suggested by experimental psychology. Since then, the number and magnitude of anomalies noticed by researchers have increased and the focus of finance academic discussion has shifted. Overall, the main arguments of behavioral finance is categorized by Shefrin (2000) as follows:

1. Financial practitioners commit errors due to relying on rules of thumb called heuristics.

2. Frame of a decision problem influences financial practitioners when making their decisions.
3. Heuristic-driven biases and framing effects affect the prices in financial markets to deviate from fundamental values.

Behavioral finance is criticized by efficient market supporters for not having any unifying principles to explain the origin of behavioral anomalies. At the same time, behavioral economists criticize efficient market supporters for making unrealistic assumptions and systematic errors in predicting behavior. Although these paradigms differ on their foundations, under some realistic assumptions they can be merged together as a more real approximation of financial markets. Discussion of a more general model of financial markets on the basis of fuzzy logic is the main purpose of this paper.

The plan of the rest of the paper is structured as follows. We first provide prerequisite materials in fuzzy logic, measures, and integrals underlying the proposed view of financial markets. Then, we present a new framework to define subjective rationality as a broad concept. Consistent with this framework, we exemplify decision situations in which different decision theories account for rational behavior. Then, we discuss how some of the well-known behavioral biases/anomalies are rationalized in the proposed framework. Specifically, we focus on "insurance and gambling", and "equity premium puzzle" for illustrative purposes. Lastly, we discuss a possibility and advantages of a general model of financial markets ("GMFM") on the basis of fuzzy logic and offer conclusions.

2 Preliminaries

Fuzzy Set Theory: The ideas of fuzzy sets and fuzzy logic date back to Black (1937) and it has been mathematically formalized by Zadeh (1965). The most common type of fuzzy sets is standard fuzzy sets. Each of the standard fuzzy sets is uniquely defined by a membership function of the form, $\mu_{\tilde{A}} : \Omega \to [0, 1]$ where Ω denotes universal set and $\tilde{A}$ is a fuzzy subset of Ω. Since, a characteristic function of crisp (classical) sets is a special case of a membership function of fuzzy sets, $\{0,1\} \subseteq [0,1]$, fuzzy sets are considered a formal generalization of classical sets.

Three basic operations on sets - complementation, intersection, and union - are not unique in fuzzy sets as they are in crisp sets. The standard complement of a fuzzy set $\tilde{A}$ is a fuzzy set $\tilde{A}^c$ with the membership function $\mu_{\tilde{A}^c} = 1 - \mu_{\tilde{A}}$. The standard intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set with the membership function $\mu_{\tilde{A}\cap\tilde{B}}(\omega) = \min\{\mu_{\tilde{A}}(\omega), \mu_{\tilde{B}}(\omega)\}$ and the standard union of two fuzzy sets is also a fuzzy set with $\mu_{\tilde{A}\cup\tilde{B}}(\omega) = \max\{\mu_{\tilde{A}}(\omega), \mu_{\tilde{B}}(\omega)\}$ where $\omega \in \Omega$.

A fuzzy set $\tilde{A}$ is said to be a subset of fuzzy set $\tilde{B}$, $\tilde{A} \subseteq \tilde{B}$, if and only if $\mu_{\tilde{A}}(\omega) \leq \mu_{\tilde{B}}(\omega)$, $\forall\omega \in \Omega$ given that fuzzy sets $\tilde{A}$ and $\tilde{B}$ are defined on the same universal set Ω.

One of the most important concepts of fuzzy sets is an α-cut of a fuzzy set which is one way of connecting fuzzy sets to crisp sets. An α-cut of a fuzzy set $\tilde{A}$ on Ω denoted as ${}^{\alpha}A$ is a crisp set that satisfies ${}^{\alpha}A = \{\omega \mid \mu_{\tilde{A}}(\omega) \geq \alpha\}$ where $\alpha \in [0,1]$. It is then easy to see that the set of all distinct α-cuts is always a nested family of crisp sets. That is, ${}^{\alpha_1}A \subseteq {}^{\alpha_2}A$ is satisfied when $\alpha_1 \geq \alpha_2$. It is also well-known that each fuzzy set is uniquely represented by the associated family of its α-cuts, $\tilde{A} = \bigcup_{\alpha\in[0,1]}\{\alpha \cdot [{}^{\alpha}A]\}$. A strong α-cut, denoted as ${}^{\alpha+}A$, is similar to the α-cut representation, ${}^{\alpha+}A = \{\omega \mid \mu_{\tilde{A}}(\omega) > \alpha\}$, but with a stronger condition. ${}^{0+}A$ and ${}^{1}A$ are called support and core of a fuzzy set $\tilde{A}$, respectively. When the core of a fuzzy set $\tilde{A}$ is not empty, ${}^{1}A \neq \emptyset$, $\tilde{A}$ is called normal, otherwise it is called subnormal. A fuzzy set is convex if and only if all its α-cuts are convex sets as in the classical sense.

Definition 1. *A fuzzy set $\tilde{A}$ on $\mathcal{R}$ (a set of real numbers) is a fuzzy number if (i) $\tilde{A}$ is a normal fuzzy set, (ii) ${}^{\alpha}A$ is a closed interval for every $\alpha \in (0,1]$ and (iii) the support of $\tilde{A}$ is bounded.*

A fuzzy number $\tilde{A}$ is said to be less than a fuzzy number $\tilde{B}$, $\tilde{A} \leq \tilde{B}$, if ${}^{\alpha}A \leq {}^{\alpha}B$ is satisfied for every $\alpha \in (0,1]$. Equivalence of two fuzzy numbers, $\tilde{A} = \tilde{B}$, is attained when $\tilde{A} \leq \tilde{B}$ and $\tilde{B} \leq \tilde{A}$. When fuzzy numbers are used to formulate linguistic concepts such as very small, small, and so on, the final constructs are called linguistic variables.

Definition 2. *Let $\mathcal{E}^n$ be a space of all fuzzy subsets of $\mathcal{R}^n$ consisting of fuzzy sets which are normal, fuzzy convex, upper semi-continuous with compact support. A fuzzy function is a mapping from universal set Ω to $\mathcal{E}^n$, $\tilde{f} : \Omega \to \mathcal{E}^n$.*

The other principle of connecting crisp sets and fuzzy sets is to fuzzify functions with an extension principle. Let f be a mapping from X to Y, $f : X \to Y$ where X and Y are crisp sets. The function f is fuzzified when it is extended to fuzzy sets on X and Y. Formally, the fuzzified function $\tilde{f}$ is a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ where $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ denote fuzzy power sets which is the family of all fuzzy subsets of X and Y, respectively. According to Zadeh's extension principle, $\tilde{B} = \tilde{f}(\tilde{A})$ for any given fuzzy set $\tilde{A} \in \mathcal{F}(X)$ and $\forall y \in Y$,

$$\tilde{B}(y) = \begin{cases} \sup\{\mu_{\tilde{A}}(x) \mid x \in X, f(x) = y\} & \text{when } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Definition 3. *Let $\tilde{A}, \tilde{B} \in \mathcal{E}^n$. If there exists $\tilde{C} \in \mathcal{E}^n$ such that $\tilde{A} = \tilde{B} + \tilde{C}$, then $\tilde{C}$ is called a Hukuhara difference $(-_h)$ of $\tilde{A}$ and $\tilde{B}$.*

Example: Let $\tilde{A}$ and $\tilde{B}$ be triangular fuzzy sets $\tilde{A} = (5, 7, 9)$ and $\tilde{B} = (1, 2, 3)$. Then, $\tilde{A} -_h \tilde{B} = (5, 7, 9) - (1, 2, 3) = (5 - 1, 7 - 2, 9 - 3) = (4, 5, 6)$. Hence, $\tilde{B} + (\tilde{A} -_h \tilde{B}) = (1, 2, 3) + (4, 5, 6) = (5, 7, 9) = \tilde{A}$. Note that, in the standard fuzzy arithmetic $\tilde{A} - \tilde{B} = (5, 7, 9) - (1, 2, 3) = (5 - 3, 7 - 2, 9 - 1) = (2, 5, 8)$ and $\tilde{B} + (\tilde{A} - \tilde{B}) = (1, 2, 3) + (2, 5, 8) = (3, 7, 11) \neq \tilde{A}$.

Definition 4. *Given a fuzzy number $\tilde{A}$ on Ω, absolute value $|\tilde{A}|$ is defined as*

$$\mu_{|\tilde{A}|}(\omega) = \begin{cases} \max\left(\mu_{\tilde{A}}(\omega), \mu_{-\tilde{A}}(\omega)\right) & \text{for } \omega \in \mathcal{R}^+, \\ 0 & \text{for } \omega \in \mathcal{R}^-. \end{cases} \tag{2}$$

Definition 5. *Given a universal set Ω and its non-empty power set $\mathcal{F}(\Omega)$ with appropriate algebraic structure, a classical measure, μ, on $\mathcal{F}(\Omega)$ is a mapping from $\mathcal{F}(\Omega)$ to a positive real number $\mathcal{R}^+$, $\mu : \mathcal{F}(\Omega) \to [0, \infty]$, with the following requirements:*

1. $\mu(\emptyset) = 0$ (vanishing at the empty set);

2. For every sequence $A_1, A_2, ...$ of pairwise disjoint sets of $\mathcal{F}(\Omega)$, if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\Omega)$ then $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (Additivity).

Note that, probability is a classical measure with $\mu(\Omega) = 1$ and $\mathcal{F}(\Omega)$ being a σ-algebra.

Definition 6. *Given a finite universal set $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ and a real-valued function $f(\omega_k)$ on Ω, the integral of f on Ω w.r.t. (with respect to) a classical measure μ is expressed as*

$$\int_{\Omega} f d\mu = \sum_{k=1}^{n} f(\omega_k)\mu(\omega_k). \tag{3}$$

Definition 7. *Given a universal set Ω and its non-empty power set $\mathcal{F}(\Omega)$ with appropriate algebraic structure, a monotone measure, η, on $\mathcal{F}(\Omega)$ is a mapping from $\mathcal{F}(\Omega)$ to a positive real number $\mathbb{R}^+$, $\eta : \mathcal{F}(\Omega) \to [0, \infty]$, with the following requirements:*[1]

1. $\eta(\emptyset) = 0$ (vanishing at the empty set);

2. For all $A, B \in \mathcal{F}(\Omega)$, if $A \subseteq B$, then $\eta(A) \leq \eta(B)$ (monotonicity);

3. If $A_1 \subseteq A_2 \subseteq \cdots A_n \in \mathcal{F}(\Omega)$, then $\lim_{n\to\infty} \eta(A_n) = \eta(\bigcup_{n=1}^{\infty} A_n)$ (continuity from below);

4. If $A_1 \supseteq A_2 \supseteq \cdots A_n \in \mathcal{F}(\Omega)$ then $\lim_{n\to\infty} \eta(A_n) = \eta(\bigcap_{n=1}^{\infty} A_n)$ (continuity from above).

Note that, 3 and 4 are trivially satisfied when Ω is finite.

Definition 8. *Given a finite universal set $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ and a non-negative (utility) function $f(\omega_k)$ on Ω where $f(\omega_1) \geq f(\omega_2) \geq ... \geq f(\omega_n)$ and $f(\omega_{n+1}) = 0$, the Choquet integral of f on Ω w.r.t. a monotone measure η is expressed as*

$$(C) \int_{\Omega} f d\eta = \sum_{k=1}^{n} (f(\omega_k) - f(\omega_{k+1}))\eta(\{\omega_1, \omega_2, ..., \omega_k\}). \tag{4}$$

[1]Monotone measures are also called fuzzy measures in the literature. Since no fuzzy sets are involved in monotone measures, following Klir (2005) we save the term "fuzzy measures" to measures defined on fuzzy sets.

Definition 9. *Let $\mathcal{Q}(\Omega)$ denote $\{(A,B) \in \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \mid A \cap B = \emptyset\}$ the set of all pairs of disjoint sets. A bi-capacity on $\mathcal{Q}(\Omega)$ is a set function $\eta : \mathcal{Q}(\Omega) \to \mathcal{R}$ that satisfies the following requirements:*[2]

1. *$\eta(\emptyset, \emptyset) = 0$;*
2. *If $A \subset B$, then $\eta(A, \cdot) \leq \eta(B, \cdot)$ and $\eta(\cdot, A) \geq \eta(\cdot, B)$.*

A bi-capacity is a generalization of capacities that enables to account for the interaction between gains and losses. η is normalized if $\eta(\Omega, \emptyset) = 1 = -\eta(\emptyset, \Omega)$.

Definition 10. *Given a finite universal set $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ and a real-valued function $f(\omega_k)$ on Ω, Choquet-like aggregation of f on Ω w.r.t. a bi-capacity $\eta(\cdot,\cdot)$ is expressed as*

$$\mathcal{C}_\eta(f) = \sum_{k=1}^{n} \big(|f(\omega_{(k)})| - |f(\omega_{(k+1)})|\big)\eta\big(\{\omega_{(1)}, ...\omega_{(k)}\} \cap N^+, \{\omega_{(1)}, ...\omega_{(k)}\} \cap N^-\big), \tag{5}$$

where $|f(\omega_1)| \geq |f(\omega_2)| \geq ... \geq |f(\omega_n)|$; $f(\omega_{n+1}) = 0$; $N^+ = \{\omega \in \Omega \mid f(\omega) \geq 0\}$ and $N^- = \{\omega \in \Omega \mid f(\omega) < 0\}$.

Example: Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the function f on Ω takes the values of $f(\omega_1) = 4$, $f(\omega_2) = 3$ and $f(\omega_3) = -2$. Then $N^+ = \{\omega_1, \omega_2\}$ and $N^- = \{\omega_3\}$,

$$\begin{aligned}\mathcal{C}_\eta(f) &= \big(|f(\omega_1)| - |f(\omega_2)|\big)\eta(\{\omega_1\}, \{\emptyset\}) \\ &\quad + \big(|f(\omega_2)| - |f(\omega_3)|\big)\eta(\{\omega_1, \omega_2\}, \{\emptyset\}) + |f(\omega_3)|\eta(\{\omega_1, \omega_2\}, \{\omega_3\}) \\ &= \eta(\{\omega_1\}, \{\emptyset\}) + \eta(\{\omega_1, \omega_2\}, \{\emptyset\}) + 2\eta(\{\omega_1, \omega_2\}, \{\omega_3\}).\end{aligned}$$

Definition 11. *Given a finite universal set $\Omega = \{\tilde{\omega}_1, \tilde{\omega}_2, ..., \tilde{\omega}_n\}$ and a fuzzy-valued function $\tilde{f}(\tilde{\omega}_k)$ on Ω, generalised Choquet-like aggregation of $\tilde{f}$ on Ω w.r.t. a fuzzy-valued bi-capacity $\tilde{\eta}(\cdot,\cdot)$is expressed as*

$$\mathcal{C}_\eta(\tilde{f}) = \sum_{k=1}^{n} \big(|\tilde{f}(\tilde{\omega}_{(k)})| -_h |\tilde{f}(\tilde{\omega}_{(k+1)})|\big)\tilde{\eta}\big(\{\tilde{\omega}_{(1)}, ...\tilde{\omega}_{(k)}\} \cap N^+, \{\tilde{\omega}_{(1)}, ...\tilde{\omega}_{(k)}\} \cap N^-\big), \tag{6}$$

[2]The same symbol, η, is used for both, monotone measures (capacities) and bi-capacities. This should not create any notational confusion since $\eta(\cdot)$ is a capacity and $\eta(\cdot,\cdot)$ is a bi-capacity. A fuzzified version of a bi-capacity $\eta(\cdot,\cdot)$ is further denoted as $\tilde{\eta}(\cdot,\cdot)$.

where $|\tilde{f}(\tilde{\omega}_1)| \geq |\tilde{f}(\tilde{\omega}_2)| \geq ... \geq |\tilde{f}(\tilde{\omega}_n)|$*;* $\tilde{f}(\tilde{\omega}_{n+1}) = 0$*;* $N^+ = \{\tilde{\omega} \in \Omega \mid \tilde{f}(\tilde{\omega}) \geq 0\}$ *and* $N^- = \{\tilde{\omega} \in \Omega \mid \tilde{f}(\tilde{\omega}) < 0\}$*.*

Definition 12. *A linguistic lottery is a linguistic random variable* $\tilde{S}$ *with known linguistic probability distribution* $\tilde{P}^l = \{\tilde{P}_1, \tilde{P}_2, ..., \tilde{P}_n\}$ *and is represented by a vector* $\tilde{L} = (\tilde{P}_1, \tilde{S}_1; \tilde{P}_2, \tilde{S}_2; ...\tilde{P}_n, \tilde{S}_n)$*.*

For a more detailed view of mathematical formalization of fuzzy set theory one can refer to Klir and Yuan (1995), topological properties of spaces of fuzzy sets to Diamond and Kloeden (1994), fuzzy measures and integrals to Grabisch, Sugeno and Murofushi (2000), capacities and Choquet integral to Choquet (1955), bi-capacities and Choquet-like aggregation to Grabisch and Labreuche (2005*a*), (2005*b*) and (2006), uncertainty based analysis of related topics to Klir (2005) and an application to decision theories to Aliev and Huseynov (2014).

3 Broad Concept of Subjective Rationality

The main goal of this section is to attempt to answer a question of what is meant by rational behavior when the decision maker is confronted with different type of information. The proposed framework is based on the fact that the right decision method changes when the specificity and reliability of the information change. Also, the considered notion of rationality is subjective in the sense of Gilboa, Maccheroni, Marinacci and Schmeidler (2010). That is to say, the decision maker (DM) cannot be convinced that (s)he is wrong in making them.

The current interpretation of rationality in economics and finance relies heavily on the subjective expected utility (SEU) axioms of Savage (1954). Simply, you are rational if you follow axioms of SEU and irrational if you don't. This definition of rationality is too narrow to capture a real life decision situation. Also, this definition contradicts what has been tentatively argued by many economists such as Keynes (1921), Knight (1921), Shackle (1949), Arrow (1951) and so on. Specifically, Knight (1921) makes a clear distinction between risk (when relative odds of the events are known) and uncertainty (when the degree of knowledge only allows us to work with estimates). Also, Arrow (1951) notes that descriptions of uncertain consequences can be classified into two major categories, those which use exclusively the language of probability distributions and those which call for some other principle, either to replace or to supplement. We agree with the need of another principle to be a supplement to a language of probability to better approximate a real life decision situation. This is due to the fact that information that decisions are based on are not only uncertain in nature, but at the same time imprecise and partially true. Using only a probabilistic approach is not sufficient to treat uncertainty, imprecision and partial truthness of information adequately. The main argument advanced by Zadeh (1978) is that imprecision of the real life information is possibilistic rather than probabilistic in nature and a fuzzy set theory is a necessary mathematical tool to deal with possibilistic uncertainty. While there has been substantial progress on modeling uncertainty probabilistically, economics as a discipline has been somewhat reluctant with some exceptions to account for the latter over the years. For this reason, we focus on how the rational behavior should be changed as the imprecision and reliability of the information change.

One of the motivations of this paper comes from Gilboa et al. (2010) who propose behavioral foundation of objective and subjective rationality. They specifically show how the Knightian decision theory of Bewley (2002) and the maxmin expected utility (MEU) of Gilboa and Schmeidler (1989) are complementary to each other in terms of defining objective and subjective rationality. The other motivation comes from Peters (2003) who outlines rationality as an application of right decision techniques to right problems or irrationality as a mismatch of the methodology and problem. Defining the right decision technique is the major issue in this context. One might reasonably ask "what is the right decision technique" and "what is the right problem". However, these questions can be dealt appropriately in certain circumstances. For example, an application of an objective probability-based decision technique to an objective problem is a right decision technique, while applying the same technique to a situation in which a decision maker has imprecise information is not a rational option. So that, expected utility theory (EUT) of Von Neumann and Morgenstern (1944) can be regarded as a right decision technique in probability theory applicable circumstances as it builds upon objective probabilities. If the asset returns follow the random walk theory, then application of EUT becomes acceptable. By generous stretch of imagination, a similar logic can be used to determine the right decision techniques for more general classes of information. The higher the generality of information and corresponding decision theories, the more paradoxes we can solve in the existing financial models. Specifically, Zadeh (2011) outlines the following classification of information based on its generality.

Numerical Information (Ground Level – 'G') - This is single valued information with exact probability, e.g.: there is 80% chance that there will be 3 % growth in Australian economy next year.

Interval-valued Information (First Level – 'F') - This is the first order uncertainty in which probability and value take intervals, e.g.: there is 75 - 85% chance that there will be 2 – 3.5% growth in Australian economy next year.

Information with second-order uncertainty (Second Level 'S') - This is partially reliable information with sharp boundaries, e.g.: there is 70 - 85% chance that there will be 1.5 - 3.5% growth in Australian economy next year and the lower probability of the given chances being reliable is 80%.

Fuzzy Information (Third Level – 'T') - This is the information with un-sharp boundaries, e.g.: there will be moderate growth in Australian economy next year.

Z-information and visual information (Z Level – 'Z') - This is partially reliable information with un-sharp boundaries and often in natural language, e.g.: it seems likely that there will be moderate growth in Australian economy next year.

The distinction between these levels can be difficult. Sometimes we can think of one upper level as the same as one below. However, there is no doubt that the degree of informativeness or specificity of information diminishes as we move away from the ground level. The former implies the latter while it is not true for the reverse. Then, we can say that the former is more specific than the latter and the latter is more general. One can think of this generalization as a subsethood relation,

$$G \subseteq F \subseteq S \subseteq T \subseteq Z,$$

but not in a strict mathematical sense of subsethood as, for example, it is not obvious to see the relation between S and T.

We do not wish to face here the question of whether or not the information is sufficiently informative to serve a particular purpose. However, using a logic similar to Peters (2003), though his representation is vague, we approximate the definition of rationality in Table 1. Individuals can be considered subjectively rational along the diagonal in this framework. That is, for each level of information class there should be a different decision theory (methodology) to account for rational behavior. Following the argument and representation in Table 1, a broad definition of rationality is given accordingly.[3]

Definition 13. *A rational man is expected to hold belief degrees that are consistent with different decision theories for different class of information.*

	Information Classes				
Method	Numerical Information	Interval-valued Information	Second-Order Uncertainty	Fuzzy Information	Z information
Decision Theory 1	Rational	-	-	-	-
Decision Theory 2	-	Rational	-	-	-
Decision Theory 3	-	-	Rational	-	-
Decision Theory 4	-	-	-	Rational	-
Decision Theory 5	-	-	-	-	Rational

Table 1: A rationality

Table 1 representation of rationality also hides a philosophical subtlety in itself. Philosophically, rationality is not a 0/1 property. Then, we can modify Table 1 to describe a degree of irrationality of a DM. For example, an irrationality of applying decision method 1 to interval-valued information and applying the same method to Z information has to be different. More

[3]In a private conversation, David Easley mentioned that the definition is way too vague. We specifically keep the vague definition in order to capture both, precise and vague aspects of reality.

specifically, the latter is more irrational than the former. The same logic can be consistently applied to the whole Table 1 as represented in Table 2.

Method	Information Classes				
	Numerical Information	Interval-valued Information	Second-Order Uncertainty	Fuzzy Information	Z information
Decision Theory 1	Rational	- 1	- 2	- 3	- 4
Decision Theory 2	-	Rational	- 1	- 2	- 3
Decision Theory 3	-	-	Rational	- 1	- 2
Decision Theory 4	-	-	-	Rational	-
Decision Theory 5	-	-	-	-	Rational

Table 2: A changing degree of irrationality

Note that, we do not adopt maximally rational, rational and minimally rational classification of Rubinstein (2001) in this framework. We set maximum level as rational and then for each level of deviation from the corresponding information level reduce 1 unit. Also, we do not confine ourselves to only maxmin expected utility model in defining subjective rationality as proposed by Gilboa et al. (2010). This in turn, enables us to differentiate irrationality of a DM as in Table 2. A changing degree of irrationality provides a novel foundation on the theory of choice under uncertainty.

Similarly, an application of the more general decision theory where the less general is sufficient to capture the given decision situation is an inefficient use of resources but not irrational. In terms of consistency of the framework, an application of the more general decision theory should give the same result as the less general one in the corresponding information class of the less general theory, nevertheless, the latter provides computational ease. In line with consistency and computational ease, there are two fundamental reasons to move from one decision theory to another. Firstly, a more general decision theory is needed if it solves, at least, one more paradox that the existing theory can not solve. Secondly, the existing decision theory becomes inconvenient (e.g., excessively complex) at some stage and it is desirable to move to a more convenient theory. The principle of replacing the existing decision theory with a more general decision theory is similar to the principle of requisite generalization in generalized information theory (GIT). Here, a generalization is also not optional, but requisite, imposed by the nature of the decision situation.

We now illustrate and discuss the ideal candidates for the right decision theory by considering 5 financial practitioners with different uncertainty who consider three alternatives (bonds - f_1, stocks - f_2 and term deposit - f_3) for a short-term investment plan. Before, it should be emphasized that rough approximations of a real life decision situation are made for the sake of conveying the main points of the paper to readers, especially in fuzzy and Z environment.

Practitioner 1

Suppose, Practitioner 1 evaluates each alternative under strong growth (s_1), moderate growth (s_2), stable economy (s_3) and recession (s_4). He notes that the following precise utilities will be achieved under each state of the economy for different acts.

	s_1	s_2	s_3	s_4
f_1	15	9	8	4
f_2	16	9	4	0
f_3	10	10	10	10

Table 3: Utilities of each act under different states

He also has perfect information about the uncertainty of 3 states with the following (subjective) probabilities: $P(s_1) = 0.5$, $P(s_2) = 0.3$, and $P(s_3) = 0.15$. He faces the question of what option to choose.

Clearly, this situation is a perfect information situation as Practitioner 1 is in the province of probability theory. For this type of simplistic information Definition 5 is an adequate measure to capture the uncertainty and Definition 6 is a right tool to calculate overall utilities of each act. In this environment, Practitioner 1 can easily calculate $P(s_4)$ and then determine his preferences as $f_1 \succ f_2 \succ f_3$ by calculating overall utilities:

$$U(f_1) = 11.6,\ U(f_2) = 11.3 \text{ and } U(f_3) = 10.$$

In (subjective) expected utility theory, choice under uncertainty is perceived as the maximization of the mathematical expectation of individual utilities w.r.t. (subjective) probabilities. If preferences of Practitioner 1 coincide with what is suggested by (SEU) EUT, then his action is perfectly justifiable and can be regarded as rational based on our framework. So that, the optimal solution for Practitioner 1 is the bonds. In what follows, we shall try to illustrate information where Definitions 5 and 6 are not directly applicable.

Practitioner 2

Suppose, Practitioner 2 also evaluates each alternative under strong growth (s_1), moderate growth (s_2), stable economy (s_3) and recession (s_4) and he also notes the same precise utilities shown in Table 3. However, he assigns the following subjective probability intervals: $P(s_1) = [0.4, 0.45]$, $P(s_2) = [0.3, 0.35]$, and $P(s_3) = [0.15, 0.20]$. He faces the question of what option to choose.

The information of Practitioner 2 is interval-valued information. For simplicity we assumed that only his probability assessments take interval values. This can easily be extended to interval-valued utilities.

Given a set $S = \{s_1, s_2, s_3, s_4\}$ and its power set $\mathcal{F}(S)$, let $I =< [l(s_i), u(s_i)] \mid i \in \mathbb{N}_4 >$ denote 4-tuples of probability intervals on $s_i \in S$ where $l(s_i)$ and $u(s_i)$ denote corresponding lower and upper probability bounds, respectively. Given the probability intervals of 3 states, we first calculate the probability interval of state s_4, $P(s_4) = [0, 0.15]$, by solving $l(s_4) = 1 - \sum_{i=1}^{3} u(s_i)$ and $u(s_4) = 1 - \sum_{i=1}^{3} l(s_i)$. Let $\mathcal{M}$ denote a convex set of probability distribution functions p on $\mathcal{F}(S)$ satisfying

$$\mathcal{M} = \{p \mid l(s_i) \leq p(s_i) \leq u(s_i), i \in \mathbb{N}_4, \sum_{s_i \in S} p(s_i) = 1\}.$$

From the probability distributions in set $\mathcal{M}$, the lower probability measure (lower prevision) is defined for all $A \in \mathcal{F}(S)$ as $\eta(A) = \inf_{p \in \mathcal{M}} \sum_{x_i \in A} p(x_i)$. It follows from this definition that lower probabilities satisfy the conditions of Definition 7 and thus are monotone measures. Then, the lower probability measure, η, are calculated[4]

[4]Note that, $\mathcal{M}$ is non-empty set if and only if, $\sum_{i=1}^{4} l(s_i) \leq 1$ and $\sum_{i=1}^{4} u(s_i) \geq$

$$\eta(A) = \max \left\{ \sum_{x_i \in A} l(x_i), 1 - \sum_{x_i \notin A} u(x_i) \right\}, \forall A \in \mathcal{F}(S). \quad (7)$$

For the sake of clarity, let us exemplify;

$$\eta\big(\{s_1, s_4\}\big) = \max\{l(s_1) + l(s_4), 1 - u(s_2) - u(s_3)\} = 0.45,$$
$$\eta\big(\{s_1, s_3, s_4\}\big) = \max\{l(s_1) + l(s_3) + l(s_4), 1 - u(s_2)\} = 0.65.$$

Following these steps, we obtain Table 4 values of lower probability measures.

States	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_1, s_2\}$
$\eta(A)$	0.4	0.3	0.15	0	0.7
States	$\{s_1, s_3\}$	$\{s_1, s_4\}$	$\{s_2, s_3\}$	$\{s_2, s_4\}$	$\{s_3, s_4\}$
$\eta(A)$	0.55	0.45	0.45	0.35	0.20
States	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_2, s_3, s_4\}$	$\{S\}$
$\eta(A)$	0.85	0.80	0.65	0.55	1

Table 4: Lower probability measures (previsions)

Based on his probability intervals, Practitioner 2 can determine his preferences as $f_1 \succ f_3 \succ f_2$ by first ordering utility values in a descending order and then aggregating overall utilities with the Choquet integral w.r.t. the lower prevision η following Definition 8. Specifically, for a given alternative, the Choquet integral based utility is computed as

$$\begin{aligned} U(f_i) = &\Big(u\big((f_i)(s_1)\big) - u\big((f_i)(s_2)\big)\Big)\eta(\{s_1\}) \\ &+ \Big(u\big((f_i)(s_2)\big) - u\big((f_i)(s_3)\big)\Big)\eta(\{s_1, s_2\}) \\ &+ \Big(\big(u(f_i)(s_3)\big) - u\big((f_i)(s_4)\big)\Big)\eta(\{s_1, s_2, s_3\}) + u\big((f_i)(s_4)\big)\eta(S) \end{aligned}$$

given that $u\big((f_i)(s_1)\big) \geq u\big((f_i)(s_2)\big) \geq u\big((f_i)(s_3)\big) \geq u\big((f_i)(s_4)\big)$. Following the same steps for f_1, f_2 and f_3, the overall utilities for each alternative are obtained as

1. Also, equation (7) is only applicable when I satisfies $\sum_{j \neq i} l(s_j) + u(s_i) \leq 1$ and $\sum_{j \neq i} u(s_j) + l(s_i) \geq 1$. These conditions are trivially satisfied when $l(s_4) = 1 - \sum_{i=1}^{3} u(s_i)$ and $u(s_4) = 1 - \sum_{i=1}^{3} l(s_i)$.

$$U(f_1) = 10.5,\ U(f_2) = 9.7 \text{ and } U(f_3) = 10,$$

which leads to the preference order of $f_1 \succ f_3 \succ f_2$.

For Practitioner 2, due to the imprecise nature of probability intervals, Definitions 5 and 6 become deficient to directly determine the optimal solution. Therefore, we first determine convex set of probability distribution functions from the given intervals and calculate lower envelope of this closed convex set as a lower probability measure. As this lower probability measures satisfy the conditions of Definition 7, the Choquet integral becomes the right tool to determine overall utilities of each act. This is essentially the Choquet Expected Utility (CEU) proposed by Schmeidler (1989) using the notion of capacities or non-additive probabilities. With the convex capacities[5], it is also well-known that CEU is the special case of the MEU under the assumption of ambiguity-aversion (see proposition 3 of Schmeidler (1986) for proof).

One point worth to note here is that the use of lower prevision is justified with the implicit assumption of full ambiguity-aversion. If the degree of ambiguity-aversion, $\alpha \in [0,1]$, in the sense of Ghirardato, Maccheroni and Marinacci (2004) is known, Practitioner 2 is subjectively rational if he applies α-MEU. This is because α-MEU is a natural generalization of MEU to account for degree of ambiguity-aversion and ambiguity-seeking.

Suppose, instead of being fully ambiguity-averse, Practitioner 2 is 70% ambiguity-averse. Then, following the steps of α-MEU, $U(f_i)$ is determined as

$$U(f_i) = \alpha \min_{P \in \mathcal{M}} \int_S u(f_i(S))dP + (1-\alpha) \max_{P \in \mathcal{M}} \int_S u(f_i(S))dP,$$

where α denotes the degree of ambiguity-aversion. $\min_{P \in \mathcal{M}} \int_S u(f_i(S))dP$ is known from the previous calculation, as MEU coincides with CEU for convex capacities. We first determine $\max_{P \in \mathcal{M}} \int_S u(f_i(S))dP$ for $i = 1, 2, 3$ and then weight minimum and maximum utilities with α and $(1-\alpha)$ respectively to determine overall utilities of Practitioner 2. The results are

$$U(f_1) = 10.8,\ U(f_2) = 10.14 \text{ and } U(f_3) = 10.$$

[5]Capacity η is convex for all events $A, B \in \mathcal{F}(S)$ if it satisfies $\eta(A \cup B) + \eta(A \cap B) \geq \eta(A) + \eta(B)$.

Therefore, changing the degree of ambiguity-aversion changes the preference order of Practitioner 2 from $f_1 \succ f_3 \succ f_2$ to $f_1 \succ f_2 \succ f_3$. In both situation, Practitioner 2 can be regarded as subjectively rational (he can not be convinced that he is wrong in his preference order).

Practitioner 3

Again for consistency, suppose Practitioner 3 also evaluates each alternative under $S = \{s_1, s_2, s_3, s_4\}$ with the same precise utilities shown in Table 3. He also assigns the same subjective interval probabilities: $P(s_1) = [0.4, 0.45]$, $P(s_2) = [0.3, 0.35]$, $P(s_3) = [0.15, 0.20]$ and $P(s_4) = [0, 0.15]$ (computed). However, this time a probability interval of $[0.7, 0.8]$ is assigned to measure an imprecise degree of confidence of the assigned probabilities. This can be considered as a reliability of the assigned probabilities. The question remains the same.

Despite its simplicity, traditional methods are also incapable of solving this problem due to the probability intervals and the second-order uncertainty imposed by the reliability of assigned probabilities. There are two approaches we can think of to proceed with this problem.

Approach 1: A plain way to address the given situation is to use the methodology of interval-valued information by overlooking the reliability of assigned probabilities. With this approach, the same preference order of Practitioner 2 applies to Practitioner 3. That is if Practitioner 3 is fully ambiguity averse $f_1 \succ f_3 \succ f_2$ holds, but with the confidence interval of $[0.7, 0.8]$. The reason for leaving the uncertainty imposed by the reliability intact can be understood by the following illustration of Shafer (1987).

Suppose, we have asked Fred if the streets outside are slippery. He replies "Yes" and we know that 80% of the time he speaks truthfully and 20% of the time he speaks carelessly, saying whatever comes into his mind. With $p_1 =$"the streets are slippery" and $p_2 =$"the streets are not slippery" propositions, Shafer derives a belief of 0.8 in proposition $\{p_1\}$ and 0.2 in $\{p_1, p_2\}$. The main point here is that, if we don't have additional information, we should not allocate the remaining 0.2 between p_1 and p_2. In our example, the Shafer's illustration suggests that there is $[0.7, 0.8]$ units of evidence supporting $\{f_1 \succ f_3 \succ f_2\}$ and $[0.2, 0.3]$ units of evidence supporting all the combinations of preference order, $\{\{f_1 \succ f_2 \succ f_3\}, \{f_1 \succ f_3 \succ f_2\}, \{f_2 \succ$

$f_1 \succ f_3\}, \{f_2 \succ f_3 \succ f_1\}, \{f_3 \succ f_1 \succ f_2\}, \{f_3 \succ f_2 \succ f_1\}\}$.

In line with Shafer's example, the first approach concludes that, $U(f_1) = 10.5$, $U(f_2) = 9.7$ and $U(f_3) = 10$ with the reliability (confidence, accuracy) of $[0.7, 0.8]$ if the Practitioner 3 is fully ambiguity averse.

Approach 2: It is easy to note that reliability and ambiguity attitude are related to each other. More specifically, there is an inverse relationship between reliability and ambiguity-aversion. As the information gets more and more unreliable a DM should become more ambiguity averse. In that sense, ambiguity-aversion α is a function of the reliability of information $\alpha = \psi(\underline{r}, \bar{r})$, where $\underline{r}$ and $\bar{r}$ denote lower and upper reliability of information. We have not been able to determine what confidence functional (ψ) would account for rational behavior. This problem is similar to the problem of which utility function makes sense and leads to beneficial outcome. For this purpose, any utility function would suffice for an agent to be rational. In that sense, any confidence functional leading to ambiguity-aversion would suffice for our purposes as well. Suppose, $\psi(r) = 1-(\underline{r}+\bar{r})/2$ for the demonstration purpose. Then, ambiguity-aversion, α, equals 0.25 and application of α-MEU results in

$$U(f_1) = 11.25,\ U(f_2) = 10.79 \text{ and } U(f_3) = 10,$$

which leads to the preference order of $f_1 \succ f_2 \succ f_3$.

So far, α-MEU is used as the most general decision theory and it suffices to account for subjective rationality under the second-order imprecise probability. However, in the fuzzy and Z-environment, due to the fundamental level dependence of human behavior α-MEU falls short of taking this dependence into account.

Practitioner 4

Consider Practitioner 4 notes the following trends under $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$ and her possible set of states is $H = \{h_1, h_2\}$ where h_1 and h_2 (non-fuzzy in this example) stands for ambiguity-aversion and ambiguity-seeking, respectively.

- $\tilde{f}_1$ will yield high income under $\tilde{s}_1$, medium income under $\tilde{s}_2$, less than medium income under $\tilde{s}_3$ and small income under $\tilde{s}_4$;

– $\tilde{f}_2$ will yield very high income under $\tilde{s}_1$, medium income under $\tilde{s}_2$, small income under $\tilde{s}_3$ and a notable loss under $\tilde{s}_4$;

– $\tilde{f}_3$ will yield approximately the same medium income in all 4 fuzzy states of economy.

Practitioner 4 also has information that $\tilde{s}_1$ will take place with a medium probability, $\tilde{s}_2$ will take place with a less than medium probability, $\tilde{s}_3$ with a small probability and $\tilde{s}_4$ with a very small probability. The probability of her ambiguity-aversion is also known to be about 70% and she is assumed to be ambiguity-seeking when she is not ambiguity-averse. The question remains the same.

Clearly, Practitioner 4 has fuzzy information. This problem is considered as the problem of decision making under possibilistic-probabilistic information and linguistic preference. At this information level, there are also two ways of dealing with the given problem.

Approach 1: The first approach is to compute a fuzzy-number-valued lower prevision and use the Choquet integral w.r.t. to the computed lower prevision to calculate the total utility values of each act. This is essentially a generalized version of Choquet expected utility (CEU) of Schmeidler (1989) and the argument advanced by Aliev, Pedrycz, Fazlollahi, Huseynov, Alizadeh and Guirimov (2012). This approach is also consistent with the previous decision theories used for Practitioners 2 and 3.

Approach 2: The second approach is more behavioral in nature. As argued, this type of information triggers psychological factors to interact with each other. Because of capturing interaction among behavioral determinants to account for the fundamental level dependence of human behavior, behavioral decision-making with combined states under imperfect information (BDMCSII) of Aliev, Pedrycz and Huseynov (2013) serves perfectly well to determine the optimal action of Practitioner 4 in this situation. Although, the first approach is also consistent with the previous decision theories, we are in favor of using BDMCSII in our framework due to an interaction among factors induced by the fuzzy environment.

BDMCSII combines fuzzy states of nature and fuzzy states of the decision maker as $\Omega = S \times H$ (cartesian product of S and H) with the elements of

$\tilde{\omega}_i^j = (\tilde{s}_i, \tilde{h}_j)$ to account for the fundamental level dependence of human behavior. Here, neither Definition 5 nor Definition 7 is adequate to capture the given uncertainty as well as the dependence of S and H. Therefore, at this level, among the fuzzy set of actions, $A = \{\tilde{f} \in A \mid \tilde{f} : \Omega \to X\}$ where X denotes a space of fuzzy outcomes, BDMCSII determines an optimal action $\tilde{f}^* \in A$ with $\tilde{U}(\tilde{f}^*) = max_{\tilde{f} \in A} \int_\Omega \tilde{U}(\tilde{f}(\tilde{\omega}))d\tilde{\eta}$ which implies that an overall utility of an action is determined by a fuzzy number valued bi-capacity based aggregation over space Ω. The step by step formulation of the given problem by BDMCSII is as follows.

Suppose the following outcomes represented by triangular fuzzy numbers corresponds to each act under different states of economy. In other words, the given fuzzy numbers are precisiated forms of the given linguistic gains.

	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{s}_4$
$\tilde{f}_1$	$(8, 11, 14)$	$(5, 8, 11)$	$(3, 6, 9)$	$(1, 3, 5)$
$\tilde{f}_2$	$(11, 15, 19)$	(5,8,11)	$(1, 3, 5)$	$(-3, -1.5, 0)$
$\tilde{f}_3$	$(5, 8, 11)$	$(5, 8, 11)$	$(5, 8, 11)$	$(5, 8, 11)$

Table 5: Fuzzy outcomes of each act under different states

We first assign fuzzy utilities $\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^j))$ (utility of action $\tilde{f}_k$ under state of economy $\tilde{s}_i$ when her own state is h_j) by applying a technique of value function of Tversky and Kahneman (1992),

$$\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^1)) = \begin{cases} (\tilde{f}_k(\tilde{s}_i))^\alpha & \text{when } \tilde{f}_k(\tilde{s}_i) \geq 0, \\ -\lambda(-\tilde{f}_k(\tilde{s}_i))^\beta & \text{when } \tilde{f}_k(\tilde{s}_i) < 0; \end{cases} \quad (8)$$

$$\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^2)) = \begin{cases} (\tilde{f}_k(\tilde{s}_i))^\beta & \text{when } \tilde{f}_k(\tilde{s}_i) \geq 0, \\ -\lambda(-\tilde{f}_k(\tilde{s}_i))^\alpha & \text{when } \tilde{f}_k(\tilde{s}_i) < 0; \end{cases} \quad (9)$$

where $\alpha = 0.88$, $\beta = 1.25$ and $\lambda = 2.25$. For instance,

$$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1)) = (\tilde{f}_1(\tilde{s}_1))^\alpha = (8^{0.88}, 11^{0.88}, 14^{0.88}) \approx (6, 8, 10),$$

$$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^2)) = (\tilde{f}_1(\tilde{s}_1))^\beta = (8^{1.25}, 11^{1.25}, 14^{1.25}) \approx (13, 20, 27).$$

A similar calculation follows for other utilities. The absolute values of approximate results in a descending order using a compatibility based ranking of fuzzy numbers are given in Table 6. Absolute values of utilities are only different at $\tilde{f}_2(\tilde{s}_4, h_1)$ and $\tilde{f}_2(\tilde{s}_4, h_2)$.

$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^2)) \approx (13, 20, 27)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^2)) \approx (4, 9, 16)$
$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1)) \approx (6, 8, 10)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^1)) \approx (3, 5, 7)$
$\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^2)) \approx (1, 4, 7)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^1)) \approx (1, 3, 4)$	
$\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^2)) \approx (20, 30, 40)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^1)) \approx (8, 11, 13)$
$\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^1)) \approx (4, 6, 8)$	$\lvert\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^1))\rvert \approx (0, 4, 9)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^2)) \approx (1, 4, 7)$
$\lvert\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^2))\rvert \approx (0, 3, 6)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^1)) \approx (1, 3, 4)$	
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_1^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_3^2)) \approx (7, 13, 20)$
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_4^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_1^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_2^1)) \approx (4, 6, 8)$
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_3^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_4^1)) \approx (4, 6, 8)$	

Table 6: Fuzzy utilities under different states of economy and decision-maker

After assigning fuzzy utilities to each act, the next step is to construct a fuzzy joint probability distribution (FJP) $\tilde{P}$ on Ω given the fuzzy marginal probabilities of $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$ and $H = \{h_1, h_2\}$. With the given imperfect information on probabilities such as medium probability, less than medium probability, and so on, the fuzzy marginal probability distributions of S and H can be represented by the following triangular fuzzy numbers.[6]

$$\tilde{P}(\tilde{s}_1) = (0.45, 0.50, 0.55),\ \tilde{P}(\tilde{s}_2) = (0.325, 0.35, 0.375),$$
$$\tilde{P}(\tilde{s}_3) = (0.1, 0.125, 0.15),\ \tilde{P}(\tilde{s}_4) = (0, 0.025, 0.125) \text{ (computed)},$$
$$\tilde{P}(h_1) = (0.65, 0.70, 0.75),\ \tilde{P}(h_2) = (0.25,\ 0.30, 0.35) \text{ (computed)}.$$

Given the fuzzy marginal probability distributions of S and H, the FJP distribution is obtained on the base of positive and negative dependence concept of Wise and Henrion (1985).[7] Formally, the FJP is obtained with,

[6]By convention, we precisiate $(n-1)$ of the given linguistic probabilities and compute the last one in order to add up total probabilities to 1.

[7]Given the numerical probabilities $P(A)$ and $P(B)$, the joint probability of A and B is $P(A, B) = P(A)P(B)$ if A and B are independent, $P(A, B) = \min\big(P(A), P(B)\big)$ if A and B are positively dependent, and $P(A, B) = \max\big(P(A) + P(B) - 1, 0\big)$ if A and B have opposite dependence. Equations (10) and (11) are the extensions of these formulations to fuzzy probabilities via α-cuts.

$$\tilde{p}(\tilde{s}_i, h_j) = \bigcup_{\alpha\in[0,1]} \alpha\big[{}^{\alpha}p_1(s_i){}^{\alpha}p_1(h_j), \min\big({}^{\alpha}p_2(s_i), {}^{\alpha}p_2(h_j)\big)\big], \tag{10}$$

$$\tilde{p}(\tilde{s}_i, h_j) = \bigcup_{\alpha\in[0,1]} \alpha\big[\max\big({}^{\alpha}p_1(s_i) + {}^{\alpha}p_1(h_j) - 1, 0\big), {}^{\alpha}p_2(s_i){}^{\alpha}p_2(h_j)\big] \tag{11}$$

for positive and negative dependence, respectively.

For $\tilde{f}_1$ and $\tilde{f}_3$, there are positive dependences between, $(\tilde{s}_1, h_1)$, $(\tilde{s}_2, h_1)$, $(\tilde{s}_3, h_1)$, $(\tilde{s}_4, h_1)$ and negative dependences between $(\tilde{s}_1, h_2)$, $(\tilde{s}_2, h_2)$, $(\tilde{s}_3, h_2)$, $(\tilde{s}_4, h_2)$, and for $\tilde{f}_2$ there are positive dependences between, $(\tilde{s}_1, h_1)$, $(\tilde{s}_2, h_1)$, $(\tilde{s}_3, h_1)$, $(\tilde{s}_4, h_2)$ and negative dependences between $(\tilde{s}_1, h_2)$, $(\tilde{s}_2, h_2)$, $(\tilde{s}_3, h_2)$, $(\tilde{s}_4, h_1)$ due to Baillon and Bleichrodt (2015). That is to say, people are ambiguity-averse in the positive domain and ambiguity-seeking in the negative domain. Then, for $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$, $\tilde{p}(\tilde{s}_1, h_1)$ is computed as follows given $\alpha = 0, 0.5, 1$.

$$\begin{aligned}\big[{}^{0}p_1(\tilde{s}_1){}^{0}p_1(h_1), \min\big({}^{0}p_2(\tilde{s}_1){}^{0}p_2(h_1)\big)\big] &= [0.45 * 0.65, \min(0.55, 0.75)]\\ &\approx [0.293, 0.55];\end{aligned}$$

$$\begin{aligned}\big[{}^{.5}p_1(\tilde{s}_1){}^{.5}p_1(h_1), \min\big({}^{.5}p_2(\tilde{s}_1){}^{.5}p_2(h_1)\big)\big] &= [0.475 * 0.675, \min(0.525, 0.725)]\\ &= [0.32, 0.525];\end{aligned}$$

$$\begin{aligned}\big[{}^{1}p_1(\tilde{s}_1){}^{1}p_1(h_1), \min\big({}^{1}p_2(\tilde{s}_1){}^{1}p_2(h_1)\big)\big] &= [0.5 * 0.7, \min(0.5, 0.7)]\\ &= [0.35, 0.5].\end{aligned}$$

Hence, $\tilde{p}(\tilde{s}_1, h_1)$ can be approximated by the trapezoidal fuzzy number of $(0.293, 0.35, 0.5, 0.55)$. For $\tilde{f}_1$ and $\tilde{f}_3$ the FJPs of $\tilde{s}_i$ and h_j are as follows

$$\begin{gathered}\tilde{p}(\tilde{s}_1, h_1) = (0.293, 0.35, 0.5, 0.55),\ \tilde{p}(\tilde{s}_2, h_1) = (0.211, 0.245, 0.350, 0.375),\\ \tilde{p}(\tilde{s}_3, h_1) = (0.065, 0.088, 0.125, 0.15),\ \tilde{p}(\tilde{s}_4, h_1) = (0, 0.018, 0.025, 0.125),\\ \tilde{p}(\tilde{s}_1, h_2) = (0, 0, 0.150, 0.193),\ \tilde{p}(\tilde{s}_2, h_2) = (0, 0, 0.105, 0.131),\\ \tilde{p}(\tilde{s}_3, h_2) = (0, 0, 0.038, 0.053),\ \tilde{p}(\tilde{s}_4, h_2) = (0, 0, 0.008, 0.044).\end{gathered}$$

The FJPs for $\tilde{f}_2$ are the same as $\tilde{f}_1$ and $\tilde{f}_3$ for all the combinations but two, due to an inverse relationship in $\tilde{s}_4$.

$\tilde{p}(\tilde{s}_4, h_1) = (0, 0, 0.008, 0.044)$, $\tilde{p}(\tilde{s}_4, h_2) = (0, 0.018, 0.025, 0.125)$.

The next step is to construct a fuzzy valued bi-capacity $\tilde{\eta}(\cdot, \cdot)$ based on the obtained FJPs. A fuzzy valued bi-capacity is defined, $\tilde{\eta}(\tilde{A}, \tilde{B}) = \tilde{\eta}(\tilde{A}) - \tilde{\eta}(\tilde{B})$, as a difference of fuzzy-valued lower probabilities $\tilde{\eta}(\tilde{A})$ and $\tilde{\eta}(\tilde{B})$.

Given a set $\Omega = \{\omega_1^1, \omega_1^2, \omega_2^1, ..., \omega_4^1, \omega_4^2\}$ and its power set $\mathcal{F}(\Omega)$, let ${}^\alpha I =< [{}^\alpha l_i, {}^\alpha u_i] \mid i \in \mathbb{N}_8 >$ denote 8-tuples of probability intervals on $\omega_i^j \in \Omega$ where ${}^\alpha l_i$ and ${}^\alpha u_i$ denote corresponding lower and upper bounds of α-cuts of the computed FJPs, respectively. Consistent with Practitioners 2 and 3, let $\tilde{\mathcal{M}}$ denote a set of fuzzy probabilities $\tilde{p}$ on $\mathcal{F}(\Omega)$ satisfying

$$\tilde{\mathcal{M}} = \{\tilde{p} \mid {}^\alpha l(\omega_i^j) \leq {}^\alpha p(\omega_i^j) \leq {}^\alpha u(\omega_i^j), i \in \mathbb{N}_4, j \in \mathbb{N}_2, \textstyle\sum_{\omega_i^j \in \Omega} \tilde{p}(\omega_i^j) = 1\}.$$

From the fuzzy probabilities in set $\tilde{\mathcal{M}}$, the lower probability measure is defined for all $\tilde{A} \in \mathcal{F}(\Omega)$ as $\tilde{\eta}(\tilde{A}) = \inf_{\tilde{p} \in \tilde{\mathcal{M}}} \sum_{x_i \in \tilde{A}} \tilde{p}(x_i)$. An α-cut of a fuzzy lower probability measure, ${}^\alpha\eta$, are calculated same as Practitioner 2

$$ {}^\alpha\eta(\tilde{A}) = \max\left\{\sum_{x_i \in \tilde{A}} {}^\alpha l(x_i), 1 - \sum_{x_i \notin \tilde{A}} {}^\alpha u(x_i)\right\}, \forall \tilde{A} \in \mathcal{F}(\Omega). \tag{12}$$

For the sake of clarity, let us exemplify;

$$ {}^\alpha\eta(\tilde{\omega}_1^2) = \max\{{}^\alpha l(\tilde{\omega}_1^2), 1 - \sum_{i \neq 1, j \neq 2} {}^\alpha u(\tilde{\omega}_i^j)\} = {}^\alpha l(\tilde{\omega}_1^2) = 0.$$

Hence,

$$\eta(\tilde{\omega}_1^2) = (0, 0, 0).$$

$$\begin{aligned}
{}^\alpha\eta(\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1) &= \max\Big\{\big({}^\alpha l(\tilde{\omega}_1^2) + {}^\alpha l(\tilde{\omega}_2^2) + {}^\alpha l(\tilde{\omega}_1^1) + {}^\alpha l(\tilde{\omega}_2^1)\big), \\
&\quad \big(1 - {}^\alpha u(\tilde{\omega}_3^1) - {}^\alpha u(\tilde{\omega}_3^2) - {}^\alpha u(\tilde{\omega}_4^1) - {}^\alpha u(\tilde{\omega}_4^2)\big)\Big\} \\
&= \big(1 - {}^\alpha u(\tilde{\omega}_3^1) - {}^\alpha u(\tilde{\omega}_3^2) - {}^\alpha u(\tilde{\omega}_4^1) - {}^\alpha u(\tilde{\omega}_4^2)\big) \\
&= 0.629 + 0.176 \cdot \alpha.
\end{aligned}$$

Hence,

$$\eta(\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1) \approx (0.63, 0.8, 0.8).$$

Based on this formulation, we obtain Table 7 on the values of $\tilde{\eta}$ for $\tilde{f}_2$. Note that, $\tilde{\eta}(B)$ should be directly set to 0 for $\tilde{f}_1$ and $\tilde{f}_3$ when calculated with the same way as there is no loss.[8]

$A, B \subset \Omega$	$\tilde{\eta}(A)$	$\tilde{\eta}(B)$	$\tilde{\eta}(A, B)$
$\{\tilde{\omega}_1^2\}, \{\emptyset\}$	(0,0,0)	(0,0,0)	(0,0,0)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2\}, \{\emptyset\}$	(0,0,0)	(0,0,0)	(0,0,0)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1\}, \{\emptyset\}$	(0.29,0.45,0.45)	(0,0,0)	(0.29,0.45,0.45)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\emptyset\}$	(0.63,0.8,0.8)	(0,0,0)	(0.63,0.8,0.8)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\tilde{\omega}_4^1\}$	(0.63,0.8,0.8)	(0,0,0)	(0.63,0.8,0.8)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1\}$	(0.68,0.84,0.84)	(0,0,0)	(0.68,0.84,0.84)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}$	(0.68,0.84,0.84)	(0,0.02,0.02)	(0.68,0.82,0.82)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2, \tilde{\omega}_3^1\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}$	(0.83,0.97,0.97)	(0,0.02,0.02)	(0.83,0.95,0.95)

Table 7: Fuzzy-valued bi-capacities for $\tilde{f}_2$

Now, we calculate fuzzy overall utilities of $\tilde{f}_2$ by a fuzzy-valued bi-capacity based aggregation over space Ω using Definition 11.

$$\begin{aligned}
\tilde{U}(\tilde{f}_1) = &\big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^2))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2\}, \{\emptyset\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^2))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2\}, \{\emptyset\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1\}, \{\emptyset\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^1))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\emptyset\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^1))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\tilde{\omega}_4^1\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^2))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^2))| -_h |\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2, \}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}\big)\\
&+ \big(|\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}\big) = (3.99, 7.49, 9.61)
\end{aligned}$$

The values of overall utilities $\tilde{U}(\tilde{f}_1) = (4, 6.92, 8.91)$ and $\tilde{U}(\tilde{f}_3) = (4.12, 6.23, 8.25)$ can also be found based on the same computing scheme. One can then rank these fuzzy numbers as $\tilde{f}_2 \succ \tilde{f}_1 \succ \tilde{f}_3$.

After obtaining the fuzzy overall utilities of each act, BDMCSII goes further and formulates the degrees of preferences among alternatives, the concept we shall not discuss here in detail. The degrees of preferences among

[8] Approaches 1 and 2 of Practitioner 4 coincide with each other when there is no loss. However, approach 2 accounts for ambiguity-seeking when there is loss as in $\tilde{f}_2(\tilde{s}_4)$.

alternatives essentially capture the vagueness of preferences of decision maker in the fuzzy environment. One can refer to the original paper for a detailed formulation of vague preferences.

Practitioner 5

Now, suppose Practitioner 5 evaluates the same alternatives under the same economic conditions $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$ and he has the same information as Practitioner 4. As opposed to Practitioner 4, he has a degree of reliability (expressed in natural language) of the given information. Specifically, he is *very sure* that each of his three actions will yield the same results as Practitioner 4. He is also *sure* about the probability assessment of Practitioner 4. The question remains the same.

Clearly, Practitioner 5 has imprecise and at the same time partially true information (Z-information). Similar to the argument of Practitioner 3, there are also two approaches of solving the optimal solution for Practitioner 5.

Approach 1: An easy way to proceed with the given problem is to use BDMCSII and overlook the reliability of the given information in the first stage. The argument set forth for Practitioner 3 with the illustration of Shafer (1987) applies here with the same logic it applied for Practitioner 3. With this approach, the resulting preferences are $\tilde{f}_2 \succ \tilde{f}_1 \succ \tilde{f}_3/\underline{sure}$.

Approach 2: The second approach uses the concept of Z-number suggested by Zadeh (2011). Formally, a Z-number is defined as an ordered pair $\hat{Z} = (\tilde{A}, \tilde{B})$ of fuzzy numbers to describe a value of a variable X. Here, $\tilde{A}$ is an imprecise constraint on values of a variable X and $\tilde{B}$ is an imprecise estimation of reliability of $\tilde{A}$. One can refer to Aliev, Huseynov, Aliyev and Alizadeh (2015) for the arithmetic of Z-numbers and to Aliev, Pedrycz, Kreinovich and Huseynov (2016) for the general theory of decisions (GTD) on the basis of a Z-number concept. The GTD uses the idea of combined states argument of BDMCSII and develops a unified decision model which subsumes most of the well-known decision theories as its special cases including BDMCSII. We refer to the original paper of GTD for the details.

The solutions are compared in Table 8. Although we specifically tried to use similar information in different classes we obtained three different scenarios out of six possible scenarios. We highlight that practitioners are subjectively rational in their preference order in each scenario.

Practitioner	**Decision Theory**	**Outcome**
Practitioner 1	EUT (SEU)	$f_1 \succ f_2 \succ f_3$
Practitioner 2	MEU (CEU)	$f_1 \succ f_3 \succ f_2$
Practitioner 2	α-MEU ($\alpha = 0.7$)	$f_1 \succ f_2 \succ f_3$
Practitioner 3 (Approach 1)	MEU (CEU)/reliability	$f_1 \succ f_3 \succ f_2/[0.7, 0.8]$
Practitioner 3 (Approach 2)	α-MEU ($\alpha = 0.25$)	$f_1 \succ f_2 \succ f_3$
Practitioner 4 (Approach 2)	BDMCSII	$f_2 \succ f_1 \succ f_3$
Practitioner 5 (Approach 1)	BDMCSII/reliability	$f_2 \succ f_1 \succ f_3$/sure

Table 8: Summary of the solutions

4 Paradoxes and Rationality

In modern economic literature, there is a lot of evidence contradicting the preference of Savage's axioms as well as the theory itself as a valid representation of rationality. The evidence ranges from Ellsberg (1961) to Kahneman and Tversky (1979). However, over the years, the compiled evidence is regarded as an irrationality of economic agents while Savage's axioms retained its normative ground in economics and finance. Hence, different paradigms such as efficient market hypothesis and behavioral finance are created. It is this dogmatic view that we aimed to address in this article using imprecision and reliability of information and existing decision theories.

It is also worth to note that, decision theory is silent on the issue of distinctions between rational and irrational beliefs. Simply, beliefs are derived from observed behavior, while there is no explanation of how the belief itself is created. There is no account of belief formation process in economic theory. The arguments so far have yielded important implications on the theory of belief formation suggested by Gilboa, Postlewaite and Schmeidler (2012). An approach on the basis of imprecision and reliability of information enables us to judge when a subjectively rational belief obeys the probability calculus and when it is less structured.

This type of rationality is not the first time introduced to the economics and finance literature. For example, the famous response of Thaler (1980) to

Friedman and Savage (1948) billiard player analogy essentially suggests that acting on the basis of prospect theory may be judged as rational. The flopping of a fish analogy of Lo (2004) suggests that the same motion (flopping) makes a fish rational in one environment (underwater) and makes it irrational in another environment (dry land). The present paper reveals the bigger picture of these environments.

The main goal set forth in this section is to briefly discuss a possibility of a more general view of financial markets based on the proposed decision-theoretic and information-based framework. In what follows, we shall rationalize some of the existing well-known paradoxes of behavioral finance with the emphasis on that these rationalizations are the achievements of the used decision theories not the framework per se.

4.1 Insurance and gambling

Buying both insurance and lottery tickets is a norm rather than an exception and it is hard to reconcile with rational decision making based on only probability calculus. Buying insurance means a DM chooses a certainty in preference to uncertainty while buying a lottery ticket suggests choosing uncertainty in preference to certainty. Friedman and Savage (1948) suggest an S-shaped utility function to rationalize this behavior and the approach is criticized by Markowitz (1952). To see how both, insurance and gambling, can be rationalized by BDMCSII, consider the following example.

Alice considers to buy fire insurance for her house. She notes *a small loss* (insurance premium) under $\tilde{s}_1$ (no fire) and *a very large gain* under $\tilde{s}_2$ (fire) if she buys the insurance ($\tilde{f}_1$). She also notes *a very large loss* under $\tilde{s}_2$, while *nothing* happens under $\tilde{s}_1$ if she does not buy the insurance ($\tilde{f}_2$). Probability of fire occurring ($\tilde{P}(\tilde{s}_2)$) is *very small*. The gains and losses of Alice under different circumstances are summarized in Table 9.

	$\tilde{s}_1$ (no fire)	$\tilde{s}_2$ (fire)
$\tilde{f}_1$ (buy)	a small loss	a very large gain
$\tilde{f}_2$ (don't buy)	no loss/gain	a very large loss

Table 9: Fire Insurance

Alice also considers to buy a lottery ticket in the hope of winning the mega jackpot. She notes *a very small loss* (ticket price) under $\tilde{s}'_1$ (not win) and *a very large gain* under $\tilde{s}'_2$ (win) if she buys a ticket ($\tilde{f}'_1$). Assume for simplicity, she feels *nothing* if she does not buy a ticket ($\tilde{f}'_2$).[9] Probability of winning ($\tilde{P}(\tilde{s}'_2)$) is *very small.* The bet is summarized in Table 10.

	$\tilde{s}'_1$ (not win)	$\tilde{s}'_2$ (win)
$\tilde{f}'_1$ (buy)	a very small loss	a very large gain
$\tilde{f}'_2$ (don't buy)	no loss/gain	no loss/gain

Table 10: A lottery ticket

Moreover, probability of her risk-aversion ($\tilde{P}(h_1)$) is approximately 70 % and she is known to be risk-seeking (h_2) when she is not risk-averse (h_1).

Indeed, with the given imperfect information, one can follow the steps of BDMCSII for each case (assign utilities, find FJPs, construct fuzzy-valued bi-capacities, and aggregate with the generalized Choquet-like aggregation) and verify that, Alice should not be ashamed of buying both a lottery ticket and fire insurance at the same time.

4.2 Equity premium puzzle

The equity premium puzzle (first noted by Mehra and Prescott (1985)) refers to the large difference between the average equity returns and average returns of a fixed interest bearing bonds. To see how equity premium puzzle can be rationalized consider the following simple example.

Bob is an (only) investor with an initial wealth of W_0 and he can invest in two assets, a risky stock with an uncertain payoff and a bond with a certain payoff. The notation is given as follows.

Asset	Quantity	Price ($)	Payoff in $s \in S$
Stock	a	p	R_s
Bond	b	1	R

[9]This can be extended to the case where for example, Alice considers a feeling of regret for not buying a lottery ticket.

Further denote $\pi(s)$ as a finitely additive probability distribution over state s. The end-of-period wealth is determined as $W_s = R_s \cdot a + R \cdot b$. Using a budget constraint, $W_0 = p \cdot a + b$, one can get the end-of-period wealth as $W_s = R \cdot W_0 + [R_s - R \cdot p] \cdot a$.

First consider Bob as EU maximizer as a benchmark case

$$U(W_1, ..., W_s) = \sum_{s \in S} \pi_s \cdot u(W_s) = \sum_{s \in S} \pi_s \cdot u(R \cdot W_0 + [R_s - R \cdot p] \cdot a).$$

Without loss of generality (w.l.o.g), for an equilibrium stock price of p_0^* with a total investment in stock, $a > 0$, and bonds, $b = 0$, Bob maximizes his total utility

$$U'(a) = \sum_{s \in S} \pi_s \cdot [R_s - R \cdot p_0^*] \cdot u'(R_s \cdot a) = 0.$$

The equilibrium condition of the benchmark case is solved for the equilibrium stock price p_0^* explicitly;

$$p_0^* = \frac{\sum_{s \in S} \pi_s \cdot R_s \cdot u'(R_s \cdot a)}{R \cdot \sum_{s \in S} \pi_s \cdot u'(R_s \cdot a)}. \tag{13}$$

The benchmark equity premium is therefore the ratio of $\tau(p_0^*) = \sum_{s \in S} \pi(s) \cdot R_s / (p_0^* \cdot R)$.

Now suppose, preferences of Bob are represented by α-MEU

$$U(W_1, ..., W_s) = \alpha \cdot \min\{u(W_1), ..., u(W_s)\} + (1 - \alpha) \cdot \max\{u(W_1, ..., u(W_s)\},$$

where α denotes ambiguity-aversion. Denote $\bar{R} = \max\{R_1, ..., R_s\}$ and $\underline{R} = \min\{R_1, ..., R_s\}$. Then, for $a > 0$, the total utility of investing in "a" amount of stock is

$$U(a) = \alpha \cdot u(R \cdot W_0 + (\underline{R} - R \cdot p) \cdot a) + (1 - \alpha) \cdot u(R \cdot W_0 + (\bar{R} - R \cdot p) \cdot a).$$

W.l.o.g, for an equilibrium stock price of p^* with a total investment in stock, $a > 0$, and bonds, $b = 0$, Bob maximizes his total utility

$$U'(a) = \alpha \cdot u'(\underline{R} \cdot a) \cdot (\underline{R} - R \cdot p^*) + (1 - \alpha) \cdot u'(\bar{R} \cdot a) \cdot (\bar{R} - R \cdot p^*) = 0.$$

Again, the equilibrium condition is solved for the equilibrium stock price p^* explicitly;

$$p^* = \frac{\alpha \cdot u'(\underline{R} \cdot a)\underline{R} + (1-\alpha) \cdot u'(\bar{R} \cdot a)\bar{R}}{\alpha \cdot R[u'(\underline{R} \cdot a) - u'(\bar{R} \cdot a)] + u'(\bar{R} \cdot a) \cdot R}. \tag{14}$$

The equity premium is the ratio of $\tau(p^*) = \sum_{s \in S} \pi(s) \cdot R_s/(p^* \cdot R)$. Now, consider first the case where Bob is a risk neutral investor, $u'(\cdot) = c$.

Case 1(a): *The equity premium will be the higher the more ambiguity-averse the investor is.*

It suffices to show that the equilibrium stock price decreases when ambiguity-aversion α increases due to an inverse relationship between the equity premium and the stock price. For $u'(\cdot) = c$ one gets a simplified equilibrium stock price as

$$p^* = \frac{\alpha \cdot (\underline{R} - \bar{R}) + \bar{R}}{R}. \tag{15}$$

Since $\underline{R} < \bar{R}$, the equilibrium stock price will be the lower and the equity premium will be the higher the more ambiguity averse the investor is.

Case 1(b): *For $\alpha > 1/2$, a sufficient condition for an equity premium $\tau(p^*)$ exceeding the benchmark $\tau(p_0^*)$ is an average return exceeding the average return of the minimum and maximum return, $\big(\underline{R} + \bar{R}\big)/2 < E_\pi[R_s]$.*

For $u'(\cdot) = c$ one gets a simplified benchmark stock price as

$$p_0^* = \frac{\sum_{s \in S} \pi_s \cdot R_s}{R} = \frac{E_\pi[R_s]}{R}. \tag{16}$$

From Eq.(13) and Eq.(14) we get,

$$p^* - p_0^* = \frac{\alpha \cdot (\underline{R} - \bar{R}) + \bar{R} - E_\pi[R_s]}{R},$$

$$\begin{aligned} p^* &= p_0^* + \frac{\alpha \cdot (\underline{R} - \bar{R}) + \bar{R} - E_\pi[R_s]}{R} \\ &= p_0^* + \frac{1}{R} \cdot \big[\alpha \cdot \underline{R} + (1-\alpha) \cdot \bar{R} - E_\pi[R_s]\big]. \end{aligned}$$

For $p^* < p_0^*$ and $\tau(p^*) > \tau(p_0^*)$ to hold, $\alpha \cdot \underline{R} + (1-\alpha) \cdot \bar{R} < E_\pi[R_s]$ must be satisfied. Then, for $\alpha = 1/2$ sufficient condition for $p^* < p_0^*$ and $\tau(p^*) > \tau(p_0^*)$ to hold is $\big(\underline{R} + \bar{R}\big)/2 < E_\pi[R_s]$. This is also a sufficient condition for $\alpha > 1/2$ following the case 1(a).

Case 1(c): *For $\alpha = 1$ (full ambiguity-aversion), $\tau(p^*)$ exceeds the benchmark $\tau(p_0^*)$.*

For $\alpha = 1$, $p^* = \underline{R}/R$ and it is less than $p_0^* = E_\pi[R_s]/R$. Then, $\tau(p^*) > \tau(p_0^*)$ follows immediately.

Now consider the second case where Bob is a risk-averse investor with a strictly decreasing marginal utility function $u'(\cdot)$.

<u>Case 2:</u> *A risk-averse investor with full ambiguity-aversion* ($\alpha = 1$) *requires an equity premium* $\tau(p^*)$ *exceeding the benchmark equity premium* $\tau(p_0^*)$ *where an investor is risk-averse with no ambiguity attitude.*

For $\alpha = 1$, $p^* = \underline{R}/R$ and $p_0^* = (1/R) \cdot \big(\sum_{s \in S} \pi_s \cdot R_s \cdot u'(R_s \cdot a) / \sum_{s \in S} \pi_s \cdot u'(R_s \cdot a) \big)$. Since $\big(\sum_{s \in S} \pi_s \cdot R_s \cdot u'(R_s \cdot a) / \sum_{s \in S} \pi_s \cdot u'(R_s \cdot a) \big) > \underline{R}$, p_0^* exceeds $p*$ and $\tau(p*)$ exceeds the benchmark equity premium $\tau(p_0^*)$.

With the example of Bob, we have only added an ambiguity attitude of the DM in the sense of Ghirardato et al. (2004) to his risk attitude and confirmed that ambiguity-aversion requires an incremental equity premium. In this context, any pessimistic attitude adds an incremental requirement for the equity premium and completes the pieces of the puzzle (see Chateauneuf, Eichberger and Grant (2007) for proof of the same propositions with neo-additive capacities, and Epstein and Schneider (2008) for additional insights on ambiguity premia in a financial context).

So deeply rooted is our commitment to EUT and SEU, that we regard such patterns as paradoxical, or irrational. As we move away from the ground level of the information many of the paradoxes, it turns out, are rationalized by more general decision theories. The idea of a consistent generalized decision framework in some such way has been "in the air" for decades.

4.3 Fuzzy representation of EMH and behavioral finance

The key to understanding the main difference between efficient market hypothesis and behavioral finance comes down to the probability and crisp sets vs. possibility and fuzzy sets. With the Table 1 representation, they sit on the two extreme sides of the table. Probability calculus lies in the foundation of EMH, while fuzzy set theory lies in the foundation of behavioral finance.[10]

[10]Peters (1996) outlines how fuzzy membership functions can be used to understand some of the prominent behavioral biases.

Since fuzzy sets are the generalized version of crisp sets, a generalization of financial market problem on the basis of fuzzy logic makes perfect sense.

In view of the presented arguments, we develop the following conjectures and their validities can be tested. Unfortunately, we have not been able to obtain a general proof of any of these propositions so far. Critical reasoning and casual empiricism are the only pillars of the proposed conjectures.

Conjecture 1. *Different opinions are the results of imprecision of the information in financial markets.*

Fact vs. opinion argument and a simple mind experiment similar to Zadeh's balls-in-box problem presented at the beginning help to understand the logic behind conjecture 1. The idea here is too simple to digest. If the market participants receive fully-reliable simple numerical information (pure fact) then it is no exaggeration to say there is a homogeneous belief in the market. However, if the received information is fuzzy (e.g: medium growth) and partially true (e.g: sure), different perceptions of the information lead to different opinions. Zadeh's balls-in-box problem is the essence of the first conjecture.

Conjecture 2. *Increasing imprecision of the information leads to "behavioral biases" in decision-making.*

Conjecture 2 is the natural extension of conjecture 1. Also, it should be emphasized that most of the biases, if not all, are rationalized toward the right side of the Table 1. The "behavioral biases" in the conjecture refer to the biases in a traditional sense.

Conjecture 3. *Increased imprecision of the aggregate information of the financial markets leads to more "behavioral anomalies"*[11] *observed in the market in a traditional sense.*

[11] In the current framework, we are not in favor of calling them as anomalies as we argue that when the imprecision of the information increases, behavioral factors becomes one's only strength to play with. All of these anomalies are called anomalies because it contradicts traditional finance and it cannot be explained (or odd to explain) with the existing probabilistic tools.

In aggregate, conjecture 2 leads to conjecture 3. We duly note the argument of Friedman (1953) and De Long, Shleifer, Summers and Waldmann (1990) response. However, conjecture 3 is slightly different from both of the arguments. Here we do not conjecture the survival of noise traders in financial markets. The conjecture is rather focusing on the imperfect information that is received by everyone and perceived differently. This essentially shows the room for behavioral finance increases as we move from the left to the right on Table 1.

These conjectures elucidates a general view of financial markets that is radically different from the presented arguments of the two main paradigms of finance. The belief formation processes at the two edges of the information classes separately support EMH and behavioral finance. Therefore, efficient and inefficient markets supported by the fundamentally different beliefs have a certain truthness degree at a point in time. More specifically, a general view of financial markets include efficient and inefficient markets as its special cases.

It seems very natural to us that the asset prices oscillate between (P') and (P) illustrated in Figure 1 leaving efficient market hypothesis, undervaluation and overvaluation as special cases of more generalized asset price dynamics of financial markets without sharp boundaries. Figure 1 subsumes 3 specific fuzzy sets, namely "undervalued market", "efficient market" and "overvalued market". The abscissa axis shows the overall price level of the market and the ordinate axis shows the membership functions of fuzzy sets ranging between "0" and "1". "0" membership to the specific set means certain price level is not the member of the set while "1" shows certain price level fully belong to that specific set. For example, at P' and P the degree of efficiency of the market is "0" and, the degree of undervaluation is "1" at P' and the degree of overvaluation is "1" at P. At a point in time, for example at P*, the degree of efficiency is 0.7 and the degree of undervaluation is 0.3. It is still far from clear what measures to be used as a proxy of imprecision and reliability of the aggregate information in financial markets and approximate the degree of membership of the proposed fuzzy representation of EMH and behavioral finance.

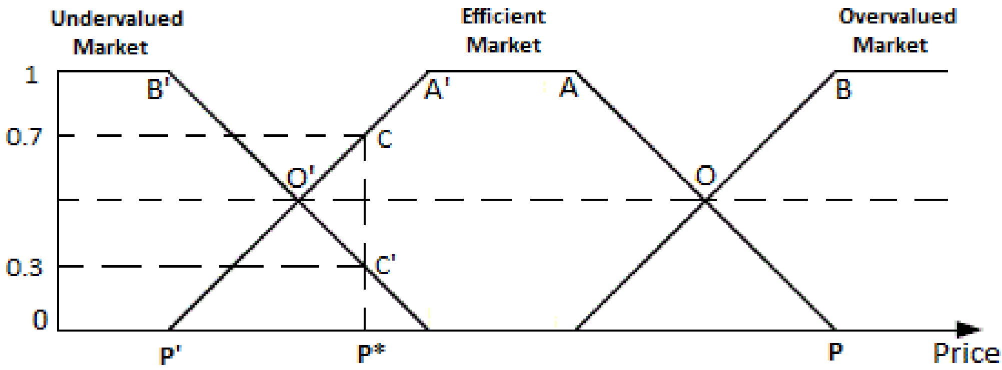

Figure 1: Representation of financial markets with fuzzy memberships

There are two main reasons that this direction to be distinguished from both paradigms. Firstly, the existence of efficient markets is not entirely excluded though it becomes an utopic concept (see Grossman and Stiglitz (1980) on the impossibility of efficient markets). Also, behavioral finance regards probability based decision-techniques as its superior though the subjective rationality presented in this paper takes behavioral factors into account when the information of the DM becomes imprecise and partially reliable.

5 Concluding Remarks

The exposition of a more general view of financial markets in the present paper shows how the existing decision theory and information science literature can be used to better understand financial markets. The main points which we have attempted to convey should now be clear. Classical probability theory favors EMH and rules out the possible room for behavioral finance. Individual behavioral "biases" and aggregate market "anomalies" mainly originate from imprecision and reliability attribute of information. Decision theories built on the foundations of a set of probabilities and in a more general setting, fuzzy set theory can be used as a complement to numeric probability based decision theories to rationalize most, if not all, of the existing behavioral "anomalies". Consequently, it will be essential to explore a more general view of financial markets by taking into account the main arguments of the present paper.

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