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TIME-VARYING BETA: A BOUNDEDLY RATIONAL EQUILIBRIUM APPROACH

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ABSTRACT. By taking into account conditional expectations and the dependence of the systematic risk of asset returns on micro- and macro-economic factors, the conditional CAPM with time-varying betas displays superiority in explaining the cross-section of returns and anomalies in a number of empirical studies. Most of the literature on time-varying beta is motivated by econometric estimation rather than explicit modelling of the stochastic behaviour of betas through agents' behaviour. Within the mean-variance framework of repeated one-period optimisation, we set up a boundedly rational dynamic equilibrium model of a financial market with heterogeneous agents and obtain an explicit dynamic CAPM relation between the expected equilibrium returns and time-varying betas. By incorporating the three most popular types of investors, fundamentalists, chartists and noise traders, into the model, we show that, independent of the fundamentals, there is a systematic change in the market portfolio, risk-return relationships, and time varying betas when investors change their behaviour, such as the chartists acting as momentum traders. In particular, we demonstrate the stochastic nature of time-varying betas and show that the commonly used rolling window estimates of time-varying betas may not be consistent with the ex-ante betas implied by the equilibrium model. The results provide a number of insights into an understanding of time-varying beta.

JEL Classification: G12, D84.

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1. INTRODUCTION

Despite the propagation of multifactor models, including Fama-French type factors, and various market anomalies, the Capital Asset Pricing Model (CAPM) remains very popular. The CAPM assumes that all investors have the same expectations about the means, variances and covariances of future returns, and hence the beta of the CAPM is assumed to be constant over time and is estimated via ordinary least squares (OLS). However, according to Bollerslev, Engle, and Wooldridge (1988), economic agents take conditional expectations of the moments of future returns and therefore these are random variables rather than constant. Due to the dependence of the systematic risk of an asset return on micro- and macro-economic factors, the assumption of beta stability has been rejected by various empirical studies over the last three decades. In fact there is strong evidence that the conditional betas are time-varying. For example, for book-to-market portfolios[1], betas of the highest decile of book-to-market stocks reached over 2.5 during the 1940s and fell to -0.5 at the end of 2001 (see for example Kothari, Shanken and Sloan (1995); Campbell and Vuolteenaho (2004) and Adrian and Franzoni (2005)). Consequently, according to Jagannathan and Wang (1996), a conditional CAPM that takes conditional expectations into account provides a convenient way to incorporate time-varying beta and displays empirical superiority in explaining the cross-section of returns and anomalies.

There exists a large literature on time-varying beta models, most of which is motivated by econometric estimation. Introduced by Engle (1982) and Bollerslev (1986), the class of GARCH models, including M-GARCH (multivariate generalized autoregressive conditional heteroskedasticity) model proposed by Bollerslev (1990), were the first to estimate time-varying betas. To model the asymmetric and nonlinear effects of beta on conditional volatility of positive and negative shocks, Braun, Nelson, and Sunier (1990) extended the basic GARCH model to an exponential GARCH (EGARCH) model. Other models include the random walk model (see, for example, Fabozzi and Francis (1978) and Collins, Ledolter, and Rayburn (1987)), the mean-reverting model (see for example Bos and Newbold (1984)), and the Markov switching models introduced in the seminal works of Hamilton ((1989), (1990)). More recently, Harvey (2001) used instrumental variables to estimate betas and showed that the estimates are very sensitive to the choice of instruments used to proxy for time-variation in the conditional betas. Among others, Campbell and Vuolteenaho (2004), Fama and French (2006), and Lewellen and Nagel (2006) assume discrete changes in betas across subsamples but constant betas within subsamples. In contrast, Ang and Chen (2007) treat betas as endogenous variables that vary slowly and continuously over time. By

[1]The book-to-market portfolios are constructed based on a book-to-market trading strategy that goes long the highest decile portfolio of stocks sorted on book-to-market ratios (value stocks) and short the lowest decile portfolio of book-to-market ratio stocks (growth stocks).

questioning the conventional wisdom that there exists strong evidence of a book-to market effect[2], Ang and Chen (2007) developed a methodology for consistently estimating time-varying betas in a conditional CAPM and found that a single-factor model performs substantially better in explaining the book-to-market premium. They demonstrated that, when betas vary over time, the standard OLS inference is misspecified and cannot be used to assess the fit of a conditional CAPM.

Economically, most models of time-varying beta are based on the representative economic agent assumption that all investors have the same subjective expectations of the means, variances and covariances of future returns. Also, most of the econometric models of time-varying beta lack any economic explanation and intuition. In the literature, the conditional CAPM with time-varying betas takes into account the conditional expectations and the dependence of the systematic risk of asset returns on micro- and macro-economic factors, but not agents' behaviour. The standard justifications for the assumption of unbounded rationality have recently been criticised and economists are giving more attention to the role of heterogeneity and bounded rationality in explaining economic phenomena. In the real world, agents can have heterogeneous subjective expectations of the means, variances and covariances of returns. Further more they are boundedly rational rather than perfectly rational. The financial markets represent the aggregation of the interaction of the boundedly rational behaviour among heterogeneous agents. Accordingly, the time-varying betas in the conditional CAPM should reflect the interaction of heterogeneous and boundedly rational agents and heterogeneity can have profound consequences for the interpretation of empirical evidence. The aim of this paper is to model explicitly the stochastic behaviour of beta by focusing on agents' heterogeneity and the resulting boundedly rational equilibrium. Different from the most of econometric models, the results in this paper provide some economic explanation and intuition of the mechanism underlying the time variation of beta.

The impact of heterogeneous beliefs among investors on the market equilibrium price has been an important focus in the literature. A number of models with investors who have heterogeneous beliefs and follow some learning processes have been previously studied[3]. Recently, using ideas from the theory of nonlinear dynamical systems, various heterogeneous agent models (HAMs) have been developed to characterize the dynamics of financial asset prices resulting from the interaction of heterogeneous agents with different attitudes towards risk and different expectations about the future evolution of asset prices. One of the key elements of this literature is the expectations feedback mechanism, see Brock and Hommes ((1997), (1998)). This framework can explain various types of market behaviour, such as the long-term swing of

[2]The book-to-market effect is that stocks with high book-to-market ratios have higher average returns than what the CAPM predicts.

[3]See, for example, Lintner (1969), Williams (1977), Huang and Litzenberger (1988), Abel (2002), Detemple and Murthy (1994), Zapatero (1998) and Basak (2000).

market prices from the fundamental price, asset bubbles, market crashes, the stylized facts and various kinds of power law behaviour[4] observed in financial markets. We refer the reader to Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2009) for surveys of recent literature on HAMs. However, most of the HAMs analysed in the literature involve a financial market with only one risky asset and are not in the context of the CAPM. More recently, some attempts have been made to develop HAMs with many assets[5]. In particular, by introducing a concept of *consensus belief*, Chiarella, Dieci and He (2010*a*, 2010*b*) show that the market equilibrium under heterogeneous beliefs can be characterized by a consensus belief, which can be constructed explicitly as a weighted average of the heterogeneous beliefs.

Within the mean-variance framework of a repeated one-period CAPM, Chiarella, Dieci and He (2010*b*) set up a framework for the CAPM with heterogeneous beliefs by considering a financial market with multiple risky assets, a riskless asset, and many heterogeneous agents. Agents having heterogeneous beliefs in the mean and variance/covariance of asset returns choose their optimal portfolio based on their beliefs. In market equilibrium, the heterogeneous beliefs are aggregated into a "consensus" belief in each period and the CAPM relation between market equilibrium returns and ex-ante aggregated beta coefficients are made explicit. This paper uses the framework developed in Chiarella, Dieci and He (2010*b*) to explore the time varying behaviour of beta. By incorporating the three most popular types of investors, fundamentalists, chartists and noise traders, into the model, we characterize the betas through the interaction of the three types of agents. It is found that the betas are time varying and they are affected by agents' behaviour. In particular, we illustrate that a change characterised by the change of a key behavioral parameter, namely, the sensitivity of chartists' predictions to recently observed returns, has a significant impact on the time-variation of betas, and hence the market portfolio and risk-return relationships between the risky assets and the market. By using the common practice of rolling OLS estimates of betas, we show that the realized betas are time-varying, but may not be consistent with the ex-ante betas implied by the equilibrium CAPM, implying that the rolling window

[4]See, for example, Day and Huang (1990), Kirman (1992), Farmer *et al.* (2004), Lux (2004), Chiarella, He and Hommes (2006), Alfarano *et al.* (2005), Gaunersdorfer and Hommes (2007), and He and Li (2007).

[5]Recent studies with many risky assets include Wenzelburger (2004), Westerhoff (2004), Böhm and Chiarella (2005), Böhm and Wenzelburger (2005), Chiarella *et al.* (2005, 2007), Westerhoff and Dieci (2006) and Horst and Wenzelburger (2008), showing that complex price dynamics may also result within a multi-asset market framework with heterogeneous beliefs. Chiarella, Dieci and He (2007) show that diversification does not always have a stabilizing role, but may act as a further source of instability in the financial market. Wenzelburger (2004) introduces a reference portfolio and Böhm and Wenzelburger (2005) show that the returns realized with an efficient portfolio do not necessarily outperform those of non-efficient portfolios. By allowing social interaction among consumers, Horst and Wenzelburger (2008) show that asset prices may behave in a non-ergodic manner.

estimates of time-varying betas can be misleading. The results provide some understanding of the economic factors underlying the time variation of beta.

The paper is organized as follows. Section 2 reviews the framework developed in Chiarella, Dieci and He ((2010*b*)). Section 3 incorporates the three most popular types of investors, fundamentalists, chartists and noise traders, into the framework and examines the steady-state equilibrium of the corresponding deterministic model. Using numerical simulation, Section 4 examines the impact of investors' behaviour on the market, including the equilibrium market prices, returns, the market portfolio, and the betas. In addition, the stochastic behaviour of the betas and the consistency of the realized betas estimated by the rolling OLS estimates are analyzed. Section 5 concludes and suggests some directions for future research. The appendix contains a result on the rescaling of the chartist parameter to different trading periods.

2. A CAPM Framework with Heterogeneous Beliefs

Consider an economy with I agents, indexed by $i = 1, 2, \cdots, I$, who invest in portfolios consisting of a riskless asset and N risky assets, indexed by $j = 1, 2, \cdots, N$, with $N \geq 1$. Let r_f be the (constant) risk free rate of the riskless asset[6] and $\widetilde{r}_j$ be the rate of return of risky asset j $(j = 1, 2, ..., N)$. Following the standard CAPM setup, we assume that agents believe that the returns of the risky assets are conditionally multivariate normally distributed. Assume that the I investors can be grouped into H agent-types, indexed by $h = 1, 2, \cdots, H$, where the agents within the same group are homogeneous in their beliefs as well as risk aversion. The constant (absolute) risk aversion coefficient of agents of type h is denoted by θ_h. We also denote by I_h the number of investors in group h and by $n_h := I_h/I$ the market fraction of agents of type h. In each period, agents update their beliefs about the first and second moment of the joint distribution of risky asset returns, and formulate their portfolio decisions in order to maximize one-period-ahead expected utility of wealth. The N-dimensional random vector of risky asset returns over the time interval from t to $t+1$ is denoted by $\widetilde{\mathbf{r}}_{t+1}$, while $E_{h,t}(\widetilde{\mathbf{r}}_{t+1})$ and $\mathbf{\Omega}_{h,t} := [Cov_{h,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})]$ indicate the conditional expectation and the conditional variance-covariance matrix of $\widetilde{\mathbf{r}}_{t+1}$ for type-h agents at time t. When agents form their beliefs at time t, their information set includes realized prices and returns up to time $t-1$. Finally, let $\mathbf{s}_t = (s_{1,t}, \cdots, s_{N,t})^T$ be the N-dimensional vector that collects the existing stock of shares at time t for each risky asset, and $\mathbf{S}_t :=$*diag*$[s_{1,t}, s_{2,t}, ..., s_{N,t}]$. Let also $\boldsymbol{\zeta}_{h,t}$ and $\boldsymbol{\zeta}_t$ be the N-dimensional vectors collecting the dollar demands of type-h agents and the aggregate dollar demand for each risky asset, respectively. The quantities $\mathbf{s}_t$, $\boldsymbol{\zeta}_{h,t}$ and $\boldsymbol{\zeta}_t$ represent average

[6]Note that when the risk-free rate is given exogenously, the net supply of the riskless asset in the market may not be zero, see Chiarella, Dieci and He (2010*b*) for the details.

amounts *per investor*. The N-dimensional vector of market clearing asset prices at time t is denoted by $\mathbf{p}_t$, while $\widetilde{\mathbf{d}}_t$ is the random vector of dividends in period t.

2.1. **The general model of boundedly rational equilibrium.** We summarize below the general form of the dynamical system which describes the market fraction multi-asset model with heterogeneous beliefs developed in Chiarella et al. (2010*b*). The heterogeneous groups of agents form their beliefs about future returns based on agents' information set at time t consisting of realized prices and returns up to time $t-1$. Such beliefs determine agents' demands, and the aggregation of such demands produces temporary equilibrium prices at time t, via market clearing. More precisely, under mean-variance preferences with *CARA* utility, the (dollar) demand vector of the risky assets at time t is given by

$$\zeta_t := \sum_{h\in H} \zeta_{h,t} = \sum_{h\in H} n_h \theta_h^{-1} \mathbf{\Omega}_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}].$$

The market clearing condition[7] at time t, $\zeta_t = \mathbf{S}_t \mathbf{p}_t$, yields the price vector $\mathbf{p}_t$ as follows

$$\mathbf{p}_t = \mathbf{S}_t^{-1} \sum_{h\in H} n_h \theta_h^{-1} \mathbf{\Omega}_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]. \tag{2.1}$$

We assume that agents' conditional expectation and conditional variance-covariance matrix of $\widetilde{\mathbf{r}}_{t+1}$are functions of realized returns, $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...$, and prices, $\mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...$, namely

$$E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...),$$
$$\mathbf{\Omega}_{h,t} = \mathbf{\Omega}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...).$$

The market clearing prices at time t, (2.1), can therefore be expressed as functions of realized returns and prices up to time $(t-1)$, $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...$, via the above specified functions $\mathbf{f}_h$ and $\mathbf{\Omega}_h$, and the same holds for random returns in period t, which depend on the random dividend in period t, $\widetilde{\mathbf{d}}_t$, as well

$$\widetilde{\mathbf{r}}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1} = \mathbf{F}(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ..., \mathbf{p}_{t-1}, \mathbf{p}_{t-2}, ...; \widetilde{\mathbf{d}}_t), \tag{2.2}$$

where $\mathbf{P}_t :=$*diag*$(p_{1,t}, p_{2,t}, ..., p_{N,t})$. Noisy dividends $\widetilde{\mathbf{d}}_t$ are assumed to follow a stochastic process that may depend, in general, on past history of prices and returns and an exogenous noise component.

In this framework it is possible to define aggregate or 'consensus' belief (see Chiarella, Dieci and He (2010*b*) for details). Given the "average" risk aversion coefficient

$$\theta_a := \Big(\sum_{h\in H} n_h \theta_h^{-1}\Big)^{-1}, \tag{2.3}$$

[7]Note that the market clearing equation will include a noisy component if noise traders are introduced into the model. See the next section.

aggregate beliefs at time t about variances/covariances and expected returns over the time interval $(t, t+1)$ are specified, respectively, as

$$\mathbf{\Omega}_{a,t} = \theta_a^{-1}\left(\sum_{h\in H} n_h \theta_h^{-1}\mathbf{\Omega}_{h,t}^{-1}\right)^{-1}, \tag{2.4}$$

$$E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) = \theta_a \mathbf{\Omega}_{a,t}\sum_{h\in H} n_h \theta_h^{-1}\mathbf{\Omega}_{h,t}^{-1}E_{h,t}(\widetilde{\mathbf{r}}_{t+1}). \tag{2.5}$$

The market clearing prices (2.1) can therefore be rewritten as if they were determined by a homogeneous agent endowed with average risk aversion θ_a and the consensus beliefs $E_{a,t}(\widetilde{\mathbf{r}}_{t+1})$, $\mathbf{\Omega}_{a,t}$, namely

$$\mathbf{p}_t = \mathbf{S}_t^{-1}\theta_a^{-1}\mathbf{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]. \tag{2.6}$$

The price reflects the equilibrium price under the market clearing condition when agents choose their optimal portfolios based on their beliefs. Therefore, we call the price as the boundedly rational equilibrium price. Such aggregation formulas are useful to derive single-period ex-ante relationships in terms of aggregate beliefs, that are formally identical to CAPM relationship, and can therefore be useful to study CAPM implications of our heterogeneous agent model. Namely, at the beginning of each time interval $(t, t+1)$ the aggregate beliefs about returns (based on information up to time $t-1$) satisfy a CAPM-like equation of the type

$$E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1} = \beta_{a,t}[E_{a,t}(\widetilde{r}_{m,t+1}) - r_f],$$

where

$$\widetilde{r}_{m,t+1} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\widetilde{\mathbf{r}}_{t+1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}$$

denotes the random return on the market portfolio of the risky assets, whereas the ex-ante "aggregate" beta coefficients are given by

$$\beta_{a,t} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]^\top \mathbf{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}]}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}].$$

Thus we see that the time variation of aggregate betas is due to agents' time varying beliefs about both the first and the second moment of the return distributions.

2.2. **Steady state equilibrium of the deterministic model.** The dynamical system (2.1) and (2.2) is stochastic in general through the dividend process and other noise terms possibly included in $\mathbf{s}_t$ (see the next section). In order to focus on the steady state of the deterministic skeleton of the model, we first assume that the amount of shares is constant over time, $\mathbf{s}_t = \mathbf{s}$. We then consider the deterministic system obtained by replacing the random dividend $\widetilde{\mathbf{d}}_t$ with the conditional expectation of the dividend process at time $t-1$, $\overline{\mathbf{d}}_t := E_{t-1}[\widetilde{\mathbf{d}}_t]$. The deterministic system thus represents the dynamics of the expectations of the equilibrium prices and returns. Denote by $\overline{\mathbf{d}}$ a steady state level of $\overline{\mathbf{d}}_t := E_{t-1}[\widetilde{\mathbf{d}}_t]$. One can easily see from (2.1) and (2.2) that in order that the deterministic system be at a steady state, stationary prices, $\overline{\mathbf{p}}$, returns, $\overline{\mathbf{r}}$, and (expected) dividends, $\overline{\mathbf{d}}$ need to satisfy[8]

$$\overline{\mathbf{p}} = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \overline{\mathbf{\Omega}}_h^{-1} [\overline{\mathbf{f}}_h - r_f \mathbf{1}], \tag{2.7}$$

where $\overline{\mathbf{\Omega}}_h := \mathbf{\Omega}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...)$, $\overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...)$, and

$$\overline{\mathbf{r}} = \overline{\mathbf{P}}^{-1} \overline{\mathbf{d}} \tag{2.8}$$

or, equivalently

$$\overline{\mathbf{p}} = \overline{\mathbf{R}}^{-1} \overline{\mathbf{d}}, \tag{2.9}$$

where[9] $\overline{\mathbf{P}}$:=*diag*$(\overline{p}_1, \overline{p}_2, ..., \overline{p}_N)$, $\overline{\mathbf{R}}$:=diag$(\overline{r}_1, \overline{r}_2, ..., \overline{r}_N)$. Equation (2.9) can be expressed component by component as

$$\overline{p}_j = \frac{\overline{d}_j}{\overline{r}_j}, \qquad j = 1, 2, ..., N,$$

which provides a representation of equilibrium prices through the usual discounted dividend formula via the appropriate rates of returns for each asset. Substitution of (2.8) (or (2.9)) into (2.7) results in a system of N equations in the equilibrium prices $\overline{p}_1, \overline{p}_2, ..., \overline{p}_N$ (or in the equilibrium returns $\overline{r}_1, \overline{r}_2, ..., \overline{r}_N$). Therefore, steady state prices, or returns, emerge endogenously from the market dynamics with evolving heterogeneous beliefs. However, in the particular case developed in the next sections, parameters will be selected in a way that steady state prices and returns are consistent with steady state agents' expectations such that

$$\overline{\mathbf{f}}_h := \mathbf{f}_h(\overline{\mathbf{r}}, \overline{\mathbf{r}}, ..., \overline{\mathbf{p}}, \overline{\mathbf{p}}, ...) = \overline{\mathbf{r}}, \quad h \in H.$$

This will highly simplify the dynamic analysis, without loss of generality. Furthermore, the 'steady state' dynamics of the noisy model in this particular case will naturally be interpreted as the standard multi-period CAPM with homogeneous beliefs and stationary beta coefficients.

3. A MODEL WITH CLASSICAL HETEROGENEOUS AGENT-TYPES

In this section we provide a representative example of the general model outlined in Section 2 with different types of beliefs and analyse the resulting dynamics for prices,

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[8]In the example of the next section, dividends will be assumed to be generated by an underlying i.i.d. process $\{\widetilde{\rho}_t\}$ for the *dividend yield*, so that $\widetilde{\mathbf{d}}_t = \mathbf{P}_{t-1}\widetilde{\rho}_t$, where $\mathbf{P}_t$:=*diag*$(p_{1,t}, p_{2,t}, ..., p_{N,t})$. It follows that $E_{t-1}[\widetilde{\mathbf{d}}_t] = \mathbf{P}_{t-1}\rho$, where $\rho := E_{t-1}[\widetilde{\rho}_t]$, and that $\overline{\mathbf{d}} = \overline{\mathbf{P}}\rho$. As an alternative specification, we may assume that dividends $\widetilde{\mathbf{d}}_t$ follow an i.i.d. process, where $E_{t-1}[\widetilde{\mathbf{d}}_t] = E[\widetilde{\mathbf{d}}_t] = \overline{\mathbf{d}}$ represents the constant (and steady state) expected dividend.

[9]Note that $\overline{\mathbf{P}}^{-1}\overline{\mathbf{p}} = \mathbf{1}$.

returns, aggregate portfolio shares and beta coefficients. This example, which is close in spirit to Chiarella, Dieci and Gardini (2005) and Chiarella, Dieci and He (2007), considers two types of agents: *fundamentalists*, who use information on the 'fundamental values', and *trend followers*, who form future return forecasts by extrapolating realized returns[10]. These two types of agents are the most common and popular ones in the literature on heterogeneous agent models. In addition, we allow for a third type of agent - *noise traders* - whose demand for each risky asset is treated as an exogenous random disturbance, described by an i.i.d. process with zero mean. The effect of noise traders is equivalent to viewing the total amount of each asset as a noisy quantity. In the following, we consider two types of agents, *fundamentalists* and *trend followers*, or *chartists*, labelled with $h = f$ and $h = c$, respectively.

We also assume that the dividend yield follows an i.i.d. process $\{\widetilde{\boldsymbol{\rho}}_t\}$, and we denote by $\boldsymbol{\rho}$ and $\mathbf{V}_\rho$ the expectation and the variance/covariance matrix of $\widetilde{\boldsymbol{\rho}}_t$, respectively. This implies that

$$\widetilde{\mathbf{d}}_t = \mathbf{P}_{t-1}\widetilde{\boldsymbol{\rho}}_t$$

where $\mathbf{P}_t$:=*diag*$(p_{1,t}, p_{2,t}, ..., p_{N,t})$, with $E_{t-1}[\widetilde{\mathbf{d}}_t] = \mathbf{P}_{t-1}\boldsymbol{\rho}$.

3.1. **Fundamentalists.** Fundamentalists compute conditional expected return of each risky asset as the sum of a constant component that represents a long-run return (depending on 'fundamental' variables), and a time varying component that accounts for the expected mean reversion towards the fundamental price. This can be expressed as

$$E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = \boldsymbol{\rho} + \alpha\mathbf{P}^{*-1}(\mathbf{p}^* - \mathbf{p}_{t-1}) = \boldsymbol{\rho} + \alpha(\mathbf{1} - \mathbf{P}^{*-1}\mathbf{p}_{t-1}),$$

where $\mathbf{p}^* = [p_1^*, p_2^*, ..., p_N^*]^\top$ is the vector of fundamental prices, $\mathbf{P}^* :=$ *diag*$[\mathbf{p}^*]$, and $\boldsymbol{\rho}$ $= [\rho_1, \rho_2, ..., \rho_N]^\top$ is the long-run component or the fundamental of asset returns. Such a 'fundamental' component is related to the dividend process, namely, $\boldsymbol{\rho}$ represents the (stationary) expected dividend yield. According to the equation above, if fundamentalists believe that a certain asset is undervalued with respect to the fundamental price, the expected return for the next trading period will include a positive capital gain and will therefore be larger than the long-run average return. We also assume that the fundamentalists have constant beliefs about the variance/covariance structure of the returns,[11]

$$\boldsymbol{\Omega}_{f,t} = \overline{\boldsymbol{\Omega}}_f.$$

[10]The chartists may or may not have information on the fundamental values, however we assume that they form their expectations based on historical prices.

[11]Fundamentalist second-moment beliefs are also related to the volatility of the dividend process. This is the perspective we adopt in the following examples.

3.2. **Trend Followers.** The trend followers are assumed to compute the expected return as a time-weighted average of observed returns, that is,

$$E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{u}_{t-1},$$

where $\mathbf{u}_{t-1}$ is a vector of sample means of past realized returns $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...$. This specification captures the extrapolative behavior of the trend followers, who expect price changes to occur in the same direction as the price trend observed over a past time window. Similarly to Chiarella Dieci and He (2007), we assume that $\mathbf{u}_{t-1}$ is computed recursively as

$$\mathbf{u}_{t-1} = \delta\mathbf{u}_{t-2} + (1-\delta)\mathbf{r}_{t-1}. \tag{3.1}$$

Effectively, the trend u_{t-1} is the average of all past historical returns $\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...$ weighted by geometric decaying weights $(1-\delta)\{1, \delta, \delta^2, \cdots\}$. Therefore, as δ decreases, the weight on the latest returns increases but decays geometrically at a rate of δ. Therefore, the trend followers can be treated as momentum traders and this is in particular the case for small δ. The variance/covariance matrix $\mathbf{\Omega}_{c,t}$ is assumed to consist of a constant component $\overline{\mathbf{\Omega}}_c$, and of a time-varying component,

$$\mathbf{\Omega}_{c,t} = \overline{\mathbf{\Omega}}_c + \lambda\mathbf{V}_{t-1},$$

where $\lambda \geq 0$ measures the sensitivity of the second-moment estimate to the sample variance $\mathbf{V}_{t-1}$. The latter is updated recursively as a function of past deviations from sample average returns using the same geometrically decaying weights as in (3.1), so that

$$\mathbf{V}_{t-1} = \delta\mathbf{V}_{t-2} + \delta(1-\delta)(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})^{\top}.$$

Note that the recursive equations for $\mathbf{u}_{t-1}$ and $\mathbf{V}_{t-1}$ can be considered as limiting cases of geometric decay processes when the memory lag length tends to infinity (see, for example, Chiarella and He (2003)).

3.3. **Noise traders.** We allow for a further class of agents, the so-called *noise traders*, whose impact is simply modelled as an additional source of random fluctuations. The demand for the risky assets (in terms of number of shares) from this type of agent at time t is described by the random vector $\widetilde{\boldsymbol{\xi}}_t := [\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t}]^{\top}$, where the $\widetilde{\xi}_{j,t}$ are assumed i.i.d. with $E(\widetilde{\xi}_{j,t}) = 0$, $j = 1, 2, ..., N$. We also assume, for the sake of simplicity, that the standard deviation of the noise trader demand for each asset is proportional to the (fixed) supply of the same asset in the market, that is, $Var(\widetilde{\xi}_{j,t}) = q^2 s_j^2$, while demands for different assets are not correlated, $E(\widetilde{\xi}_{j,t}, \widetilde{\xi}_{k,t}) = 0$ for $j, k = 1, 2, ..., N$. The parameter $q \geq 0$, that will be assumed contant for all assets, represents an additional behavioral parameter of our model, capturing the 'volatility intensity' of noise-trading. Set $\widetilde{\mathbf{\Xi}}_t :=$*diag*$(\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t})$. The market clearing condition in the presence of noise traders thus becomes, in general

$$\theta_a^{-1}\mathbf{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}] + \widetilde{\mathbf{\Xi}}_t\mathbf{p}_t = \mathbf{S}\mathbf{p}_t$$

and the market clearing prices (2.6) are therefore rewritten as

$$\mathbf{p}_t = (\mathbf{S}-\widetilde{\mathbf{\Xi}}_t)^{-1}\theta_a^{-1}\mathbf{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}], \tag{3.2}$$

where $\mathbf{\Omega}_{a,t}$ and $E_{a,t}(\widetilde{\mathbf{r}}_{t+1})$ represent the consensus beliefs defined by (2.4) and (2.5) in Section 2. Note that the introduction of noise traders does not cause the model to depart from the general setup (2.1)-(2.2) with mean-variance investors, but it is simply formally equivalent to assuming a noisy supply vector $\widetilde{\mathbf{s}}_t = \mathbf{s}-\widetilde{\boldsymbol{\xi}}_t$.

3.4. **The complete dynamic model.** We denote by θ_f and θ_c the risk aversion coefficients of the two agent-types, and by n_f and $n_c = 1-n_f$ their market fractions. Using (2.3), the market risk aversion is given by $\theta_a = \left(n_f\theta_f^{-1} + n_c\theta_c^{-1}\right)^{-1}$. From (2.4) and (2.5), the aggregate variances/covariances and expected returns are given, respectively, by

$$\mathbf{\Omega}_{a,t} = \theta_a^{-1}\left(\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\right)^{-1} = \left(\frac{n_f}{\theta_f} + \frac{n_c}{\theta_c}\right)\left(\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\right)^{-1},$$

$$\begin{aligned}E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) &= \theta_a\mathbf{\Omega}_{a,t}\left[\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1}E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}E_{c,t}(\widetilde{\mathbf{r}}_{t+1})\right]\\ &= \left(\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\right)^{-1}\left\{\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1}\left[\boldsymbol{\rho} + \alpha(\mathbf{1} - \mathbf{P}^{*-1}\mathbf{p}_{t-1})\right] + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\mathbf{u}_{t-1}\right\}.\end{aligned}$$

With the above-specified agent-types, the general dynamic model given by (2.1) and (2.2) specialises to the following noisy nonlinear dynamical system:

$$\begin{aligned}\mathbf{p}_t = (\mathbf{S}-\widetilde{\mathbf{\Xi}}_t)^{-1}\Bigg\{&\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1}\left[\boldsymbol{\rho} + \alpha(\mathbf{1} - \mathbf{P}^{*-1}\mathbf{p}_{t-1})\right] + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\mathbf{u}_{t-1}\\ &- \left(\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\mathbf{\Omega}_{c,t}^{-1}\right)r_f\mathbf{1}\Bigg\},\end{aligned} \tag{3.3}$$

$$\widetilde{\mathbf{r}}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1} = \mathbf{P}_{t-1}^{-1}\mathbf{p}_t + \widetilde{\boldsymbol{\rho}}_t - \mathbf{1}, \tag{3.4}$$

where $\mathbf{\Omega}_{c,t} = \overline{\mathbf{\Omega}}_c + \lambda\mathbf{V}_{t-1}$, and where $\mathbf{u}_{t-1}$ and $\mathbf{V}_{t-1}$ are updated according to

$$\mathbf{u}_t = \delta\mathbf{u}_{t-1} + (1-\delta)\mathbf{r}_t, \tag{3.5}$$

$$\mathbf{V}_t = \delta\mathbf{V}_{t-1} + \delta(1-\delta)(\mathbf{r}_t - \mathbf{u}_{t-1})(\mathbf{r}_t - \mathbf{u}_{t-1})^\top. \tag{3.6}$$

3.5. **The steady state of the deterministic model.** To obtain a steady-state equilibrium which is consistent with the fundament return, we consider the deterministic skeleton of the dynamical system (3.3)-(3.6) by letting $\widetilde{\boldsymbol{\rho}}_t \equiv \boldsymbol{\rho}$ and $\widetilde{\mathbf{\Xi}}_t \equiv \mathbf{0}$ for all t. Then we obtain a deterministic steady state solution $(\overline{\mathbf{p}}, \overline{\mathbf{r}}, \overline{\mathbf{u}}, \overline{\mathbf{V}})$ must necessarily satisfy the set of algebraic equations

$$\overline{\mathbf{r}} = \overline{\mathbf{u}} = \boldsymbol{\rho},$$

$$\overline{\mathbf{V}} = \mathbf{0},$$

$$\begin{aligned}\overline{\mathbf{p}} &= \mathbf{S}^{-1}\left\{\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1}\left[\boldsymbol{\rho} + \alpha(\mathbf{1} - \mathbf{P}^{*-1}\overline{\mathbf{p}})\right] + \frac{n_c}{\theta_c}\overline{\boldsymbol{\Omega}}_c^{-1}\boldsymbol{\rho} - \left(\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\overline{\boldsymbol{\Omega}}_c^{-1}\right) r_f\mathbf{1}\right\} \\ &= \mathbf{S}^{-1}\left\{\left(\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\overline{\boldsymbol{\Omega}}_c^{-1}\right)(\boldsymbol{\rho} - r_f\mathbf{1}) + \alpha\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1}(\mathbf{1} - \mathbf{P}^{*-1}\overline{\mathbf{p}})\right\}. \end{aligned} \tag{3.7}$$

The last condition represents a system of linear equations in equilibrium prices. In the following we assume that parameters are such that fundamentalist beliefs about long-run prices, $\mathbf{p}^*$, are consistent with the model steady state. If this property were not to hold, at the steady state the fundamentalists would expect a price correction that never takes place, and the average realized returns would be systematically different from long-run expected returns $\boldsymbol{\rho}$. Put differently, we assume that fundamentalists are rational in the sense that they have learnt steady state prices and returns and regard such quantities as fundamental prices. We therefore assume $\overline{\mathbf{p}} = \mathbf{p}^*$. By substituting $\overline{\mathbf{p}} = \mathbf{p}^*$ into (3.7), it turns out that $\mathbf{p}^* = \overline{\mathbf{p}}$ is determined as

$$\mathbf{p}^* = \mathbf{S}^{-1}\left(\frac{n_f}{\theta_f}\overline{\boldsymbol{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\overline{\boldsymbol{\Omega}}_c^{-1}\right)(\boldsymbol{\rho} - r_f\mathbf{1}). \tag{3.8}$$

The analytical study of the local asymptotic stability of the deterministic steady state is difficult, due to the large dimension of the system and number of parameters. Numerical simulation of (the deterministic skeleton of) the model suggests the possibility that the steady state becomes unstable via a Neimark-Sacker bifurcation when the decay parameter δ falls below a certain threshold[12]. This will be shown numerically through an example with three risky assets in the next section. We will focus on the impact of the decay parameter δ on the market, in particular the time-varying betas.

4. A NUMERICAL ANALYSIS OF TIME-VARYING BETA

Based on the model developed in the previous section, we consider numerically an example with three risky assets and a riskless asset, in the stylized market populated by fundamentalists, trend followers and noise traders. A common parameter setting is used in all the following numerical experiments, namely, $\theta_f = \theta_c := \theta = 0.005$, $r_f = 0.01$, $\boldsymbol{s} = (1, 1, 1)^T$. For simplicity, agents are assumed to be homogeneous

[12]Alternatively, such a bifurcation may occur when δ is not too high and fundamentalist proportion n_f or fundamentalist expected mean reversion α become small enough. The effect of λ is less clear and it does not seem to affect local stability of the steady state (since it is not associated with linearized terms), but increasing values of λ seem to be associated with more and more irregular fluctuations once the steady state is destabilized.

with regard to their risk aversion[13] and (the fixed part of) their variance/covariance matrix $\overline{\mathbf{\Omega}}_c = \overline{\mathbf{\Omega}}_f := \overline{\mathbf{\Omega}}$. Also, in order to focus on the correlation emerging from the endogenous dynamics of asset prices, we assume that the correlations among the three assets in $\overline{\mathbf{\Omega}}$ are zero, that is

$$\overline{\mathbf{\Omega}}_f = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2)$$

where $\sigma_1 = 0.095$, $\sigma_2 = 0.105$, $\sigma_3 = 0.11$. The fundamentalist beliefs about first moment of long-run returns are given by $\boldsymbol{\rho} = (0.06, 0.075, 0.09)^T$.

As discussed above, the fundamental prices $\mathbf{p}^*$ are assumed to be consistent with model equilibrium so that to satisfy

$$\mathbf{p}^* = \mathbf{S}^{-1}\left(\frac{n_f}{\theta_f}\overline{\mathbf{\Omega}}_f^{-1} + \frac{n_c}{\theta_c}\overline{\mathbf{\Omega}}_c^{-1}\right)(\boldsymbol{\rho} - r_f\mathbf{1}) = \frac{1}{\theta}\,\mathbf{S}^{-1}\overline{\mathbf{\Omega}}^{-1}(\boldsymbol{\rho} - r_f\mathbf{1}),$$

Dividend yields are i.i.d. normally distributed, uncorrelated across assets, and their variance/covariance matrix $\mathbf{V}_\rho$ is consistent with the exogenous part of agents' second-moment beliefs, namely, $\mathbf{V}_\rho = \overline{\mathbf{\Omega}}$. In particular, in our example the fundamental prices $\mathbf{p}^*$ turn out to be:

$$\mathbf{p}^* = \begin{bmatrix} p_1^* \\ p_2^* \\ p_3^* \end{bmatrix} = \begin{bmatrix} 1108.03 \\ 1179.14 \\ 1322.31 \end{bmatrix},$$

Note that we are assuming no correlation in the stochastic processes of the dividend yields and, accordingly, zero correlation in the fixed part of agents' second-moment. The reason for our choice is that we want to focus only on the correlation patterns emerging from the endogenous dynamics of asset prices. The parameters α, δ, λ, and n_f will possibly vary across examples, as well as the parameter q that represents the standard deviation of noise traders percentage impact.

The numerical values of parameters $\boldsymbol{\rho}$, r_f, σ_1, σ_2, σ_3 given above are interpreted as annualised parameter values. Corresponding monthly, weekly, daily parameters are obtained by rescaling annualised parameters in a standard way, according to the frequency $K = 12$ (monthly), $K = 50$ (weekly), $K = 250$ (daily). The annual expected returns are converted via the factor $1/K$ and the standard deviations are rescaled via the factor $1/\sqrt{K}$. Also the fundamentalist mean reversion parameter α is rescaled by dividing it by K, whereas parameter δ (related to the memory length) is converted using a specific rule (see Appendix). With some abuse of notation, in (3.3)-(3.6) we adopt the same symbols to denote also the rescaled parameters. A small amount of noise trading will also be assumed in some examples, namely, (normally distributed) noise on the supply of each asset will be introduced, with no correlation across assets, with standard deviation that varies across examples.

[13]Note from (3.3) that if we assume homogeneous risk aversion coefficients, what matters for the (deterministic) dynamics is the aggregate parameter (vector) $\theta\mathbf{s}$.

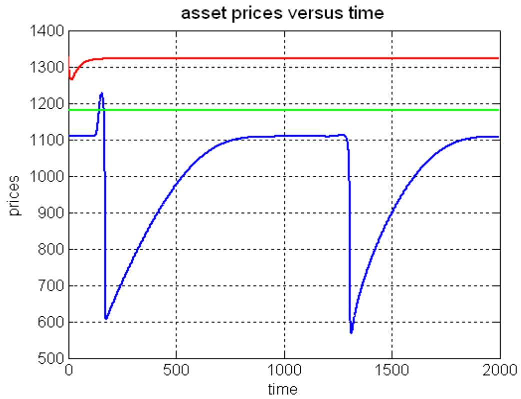
4.1. **Deterministic Dynamics.** Intuitively, when the trend followers extrapolate the recent trend in returns strongly (corresponding to a low δ), the market tends to be destabilized. To verify this effect, we first consider the changes in the equilibrium prices of the deterministic model when δ changes. In Fig. 4.1, we choose $\alpha = 0.15$, $n_f = 0.3$, $\lambda = 1.5$ and δ is decreased from $\delta = 0.88$ to $\delta = 0.84$ (at an annual frequency). These parameters are rescaled to weekly (with a period of length $1/K$ and $K = 50$). Through numerical simulations, it is found that the steady state equilibrium loses its stability when δ falls below a certain bifurcation value $\widehat{\delta}_A \in (0.879, 0.88)$, which corresponds to a Neimark-Sacker bifurcation. The six panels in Fig. 4.1, where we use blue, green and red to represent asset 1, 2, and 3, respectively, represent prices (left panels) and returns (right panels) of the risky assets for $\delta = 0.865$, $\delta = 0.855$, $\delta = 0.84$, respectively. The initial condition is selected at the steady state, except for a small deviation of the initial price of asset 3 ($p_{0,3} = 1.005 \times p_3^*$).

At first, for δ just below the bifurcation value $\widehat{\delta}_A$, only one among the three risky assets (asset 1), displays fluctuations around the steady state equilibrium level, due to the interaction of the strong extrapolation of the trend followers and mean-reverting activity of the fundamentalists. As δ decreases further, namely as the trend followers extrapolate the recent returns ever more strongly, the figure shows that two assets and finally all three assets are destabilised[14]. More precisely, further bifurcation values $\widehat{\delta}_B$ and $\widehat{\delta}_C$ for parameter δ exists, where $\widehat{\delta}_B \in (0.863, 0.864)$ and $\widehat{\delta}_C \in (0.848, 0.849)$, such that for $\widehat{\delta}_C < \delta < \widehat{\delta}_B$ also the price and return of asset 2 fluctuate around their steady state values, whereas for $\delta < \widehat{\delta}_C$ all three assets display price and return fluctuations[15]. This dependence of the price dynamics on the decay parameter δ underlines the nature of the time-varying betas of the stochastic model discussed in the following subsections.

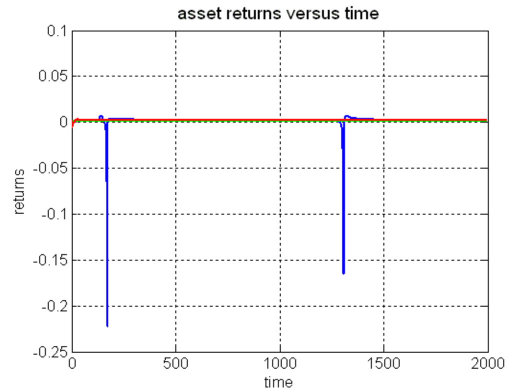
4.2. **A benchmark case of the standard 'stationary' CAPM.** The standard CAPM with homogeneous and constant beliefs can be obtained as a special case of our dynamic model, by setting $\alpha = 0$ and $\delta = 1$ and taking the initial conditions $\mathbf{u}_0 = \boldsymbol{\rho}$ and $\mathbf{V}_0 = \mathbf{0}$. Correspondingly, the expected return of the fundamentalists and the chartists are given by $E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) = \boldsymbol{\rho}$ for all t and $\overline{\boldsymbol{\Omega}}_{c,t} = \overline{\boldsymbol{\Omega}}_c = \overline{\boldsymbol{\Omega}}$. We also assume that noise trading is absent by setting $q = 0$. Under these assumptions, all agents have in fact the same fixed belief, which is fully consistent with market dynamics. As a

[14]As can be argued from Fig. 4.1, and reported by related studies (Chiarella et al. (2005, 2007)), in such multi-asset models the attractor developing from the local bifurcation may be entirely located, initially, in a lower dimensional 'manifold' of the phase space. As a consequence, we may observe systematic fluctuations of prices and returns of some assets, while other assets still remain at their steady state levels.

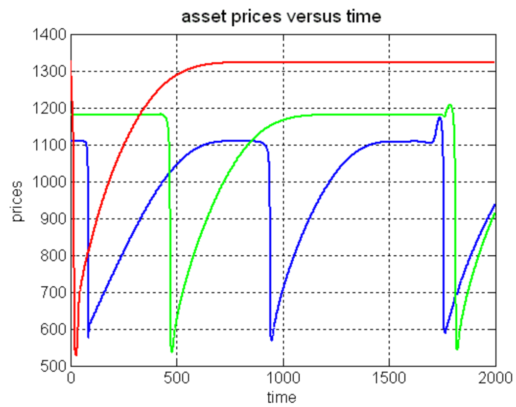
[15]For different selections of risk and return parameters, it is possible to observe qualitatively similar situations where asset 2, or asset 3 is the 'first' to be destabilized. Numerical simulation reveals that this may depend in a quite complicated way on risk-return tradeoffs.



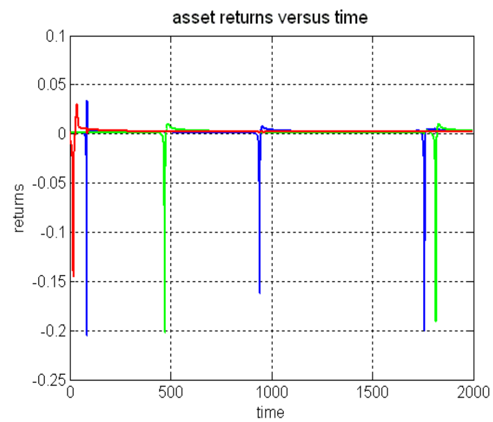
(a) $\delta = 0.865$ Equilibrium prices



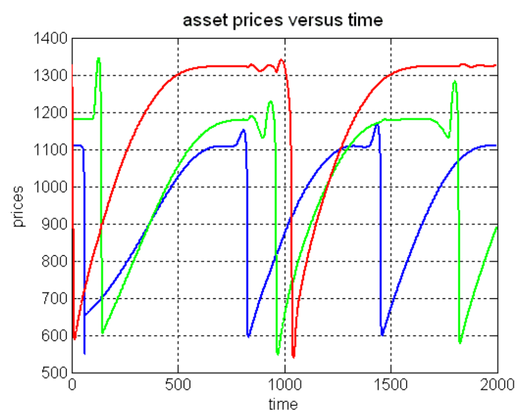
(b) Equilibrium returns of the risky assets



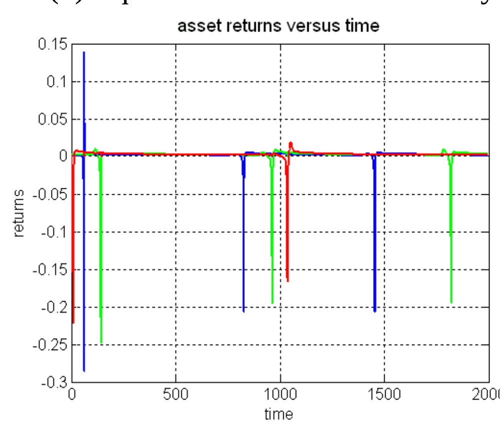
(c) $\delta = 0.855$ Equilibrium prices



(d) Equilibrium returns of the risky assets



(e) $\delta = 0.840$ Equilibrium prices



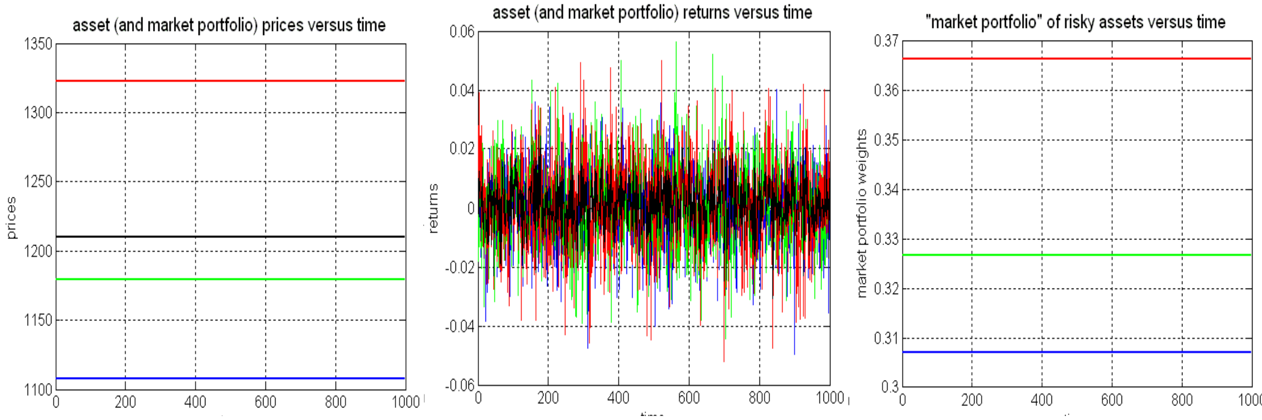
(f) Equilibrium returns of the risky assets

FIGURE 4.1. The fluctuations of prices (left panels) and returns (right panels)in the deterministic model. Here $\alpha = 0.15$, $n_f = 0.3$, $\lambda = 1.5$, whereas $\delta = 0.865$ (a,b), $\delta = 0.855$ (c,d) and $\delta = 0.840$ (e,f). The blue, green and red lines represent asset 1, 2, and 3, respectively.

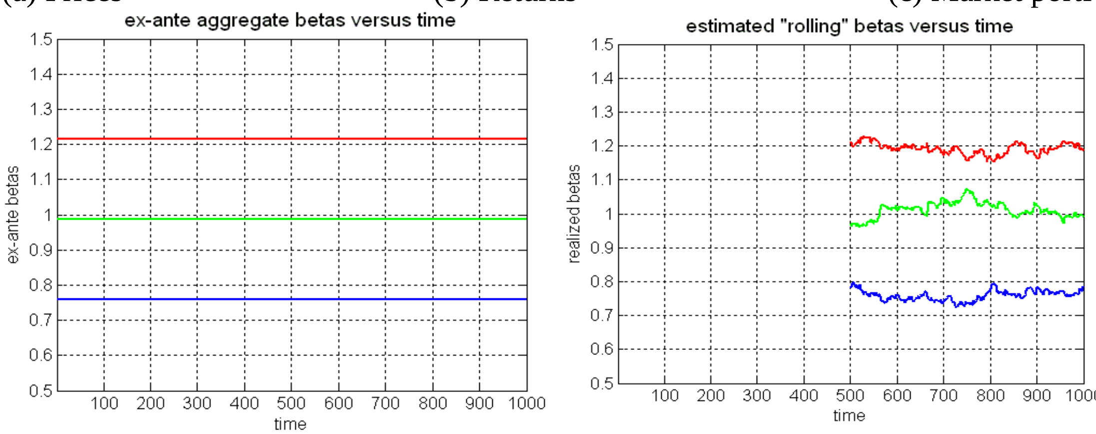
matter of fact, the dynamical system (3.3)-(3.6) then becomes[16]

$$\begin{cases} \mathbf{p}_t & = \frac{1}{\theta}\,\mathbf{S}^{-1}\overline{\mathbf{\Omega}}^{-1}(\boldsymbol{\rho} - r_f\mathbf{1}) = \mathbf{p}^*, \\ \widetilde{\mathbf{r}}_t & = \mathbf{P}^{*-1}(\mathbf{p}^* + \mathbf{P}^*\widetilde{\boldsymbol{\rho}}_t) - \mathbf{1} = \widetilde{\boldsymbol{\rho}}_t, \\ \mathbf{u}_t & = \boldsymbol{\rho}, \\ \mathbf{V}_t & = \mathbf{0}. \end{cases}$$

[16]The dynamical system turns out to be independent also of parameters n_f and λ.



(a) Prices (b) Returns (c) Market portfolio



(d) Ex-ante betas (e) Rolling estimates of betas

FIGURE 4.2. The dynamics of the benchmark stationary CAPM without noise traders. (a) The equilibrium prices of the risky assets and value of the market portfolio; (b) the equilibrium returns of the risky assets and the market portfolio; (c) the market portfolio proportions; (d) the ex-ante betas of the risky assets; and (e) the rolling estimates of the betas. Here $\alpha = 0, \delta = 1, K = 50$ and $q = 0$. The blue, green and red lines represent assets 1, 2, and 3, respectively.

This implies that prices are constant over time at their steady state fundamental level $\mathbf{p}^*$, whereas returns follow an i.i.d. random process with first and second moment[17] given, respectively, by

$$\mathbb{E}(\widetilde{\mathbf{r}}_{t+1}) = \boldsymbol{\rho}, \qquad \mathbb{V}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{V}_\rho.$$

A typical simulation of the benchmark scenario, with a weekly time step $K = 50$ and the length of the simulation $T = 1000$ time periods, is illustrated in Fig. 4.2. Apart from blue, green and red used for asset 1, 2, and 3, respectively, we use black for the market portfolio. Among the plots, Figs 4.2 (a) and (b) represents the equilibrium

[17]The latter are therefore consistent with the homogeneous and exogenous part $\overline{\boldsymbol{\Omega}}$ of agents' second moment beliefs, since we have assumed $\mathbf{V}_\rho = \overline{\boldsymbol{\Omega}}$.

prices and returns of the risky assets, respectively, demonstrating the constant equilibrium prices and i.i.d returns generated from the i.i.d. dividend processes. Fig. 4.2 (c) represents the constant proportions of the risky assets in the market portfolio. Fig. 4.2 (d) plots the constant ex-ante aggregate betas resulting from the simulation. Ex-post betas of the three assets, estimated via 'rolling' regression[18] of realized returns against market return, using a rolling windows of 500 periods, appear to fluctuate randomly around their constant ex-ante beta levels. One can see that, apart from some small random fluctuations, the rolling window estimates of the betas are consistent with the constant ex-ante betas implied by the market equilibrium.

A comment on the simulation above is in order. In our simplified setup prices are constant in the benchmark case of 'standard' CAPM, and the stationary random returns are entirely due to the normally distributed dividend yields. Due to their low average values and relatively high volatility (especially at daily and weekly frequencies), the simulation often results in negative dividend yields. Although this is not realistic[19], this experiment should be regarded as purely illustrative of a reference case in which the conditional distribution of the stationary i.i.d. return process is fully consistent with fixed and homogeneus agents' beliefs. Such a drawback could be avoided within a more rich setup, by modeling the 'benchmark' CAPM case as one where prices follow a random walk (the volatility of which is also incorporated in agents' 'steady state' beliefs), in a way that the dividend yield is responsible for only a small portion of return volatility[20].

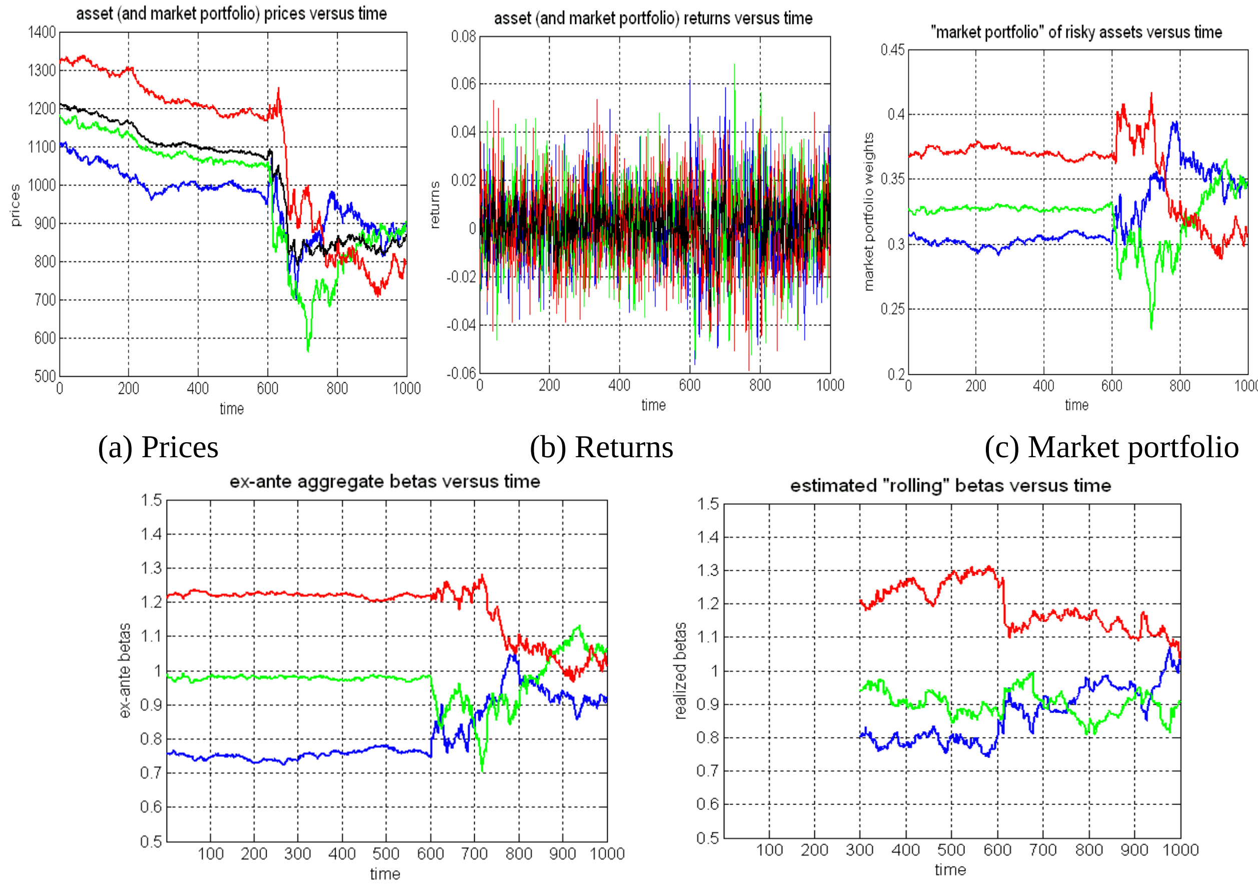
In the following examples we deviate from such a steady state scenario, interpreted as the standard 'stationary' CAPM, and show how the interaction of noise with the underlying nonlinear deterministic structure may affect significantly the risk-return relationships over time. We focus mainly on the impact of a key 'behavioral' parameter, namely, memory parameter δ, where $(1 - \delta)$ represents the weight given by trend followers to the most recent price movement.

4.3. **Trend following and time-varying betas.** We now examine the impact of changing behaviour on the betas in order to explore the nature of time-varying betas. In the

[18]Note that a common practice in empirical work involving 'rolling' betas, is represented by rolling OLS estimation over 60 months (5 years), though different choices can be found in the literature. In particular, the use of monthly returns appears to reduce the impact of transaction costs. Such issues are irrelevant within our stylized model. Therefore, we may use higher frequencies and time windows of different length, to illustrate the time-varying nature of the betas, and to emphasize the impact of model parameters on their dynamic patterns. Note also that a general feature of the rolling betas is a slow time variation, not necessarily mean-reverting to some fixed level, but not monotonic either.

[19]Unless we regard the dividend yield as including a carrying cost, proportional to the value of the asset, or we interpret a negative dividend payment as new equity issue.

[20]In such a setup, negative draws from the dividend yield process could be realistically truncated at zero, without substantially affecting the dynamics. A similar observation holds for the current model too, in the parameter regimes for which price fluctuations are large enough as compared to dividend yield volatility, for instance when parameter δ is sufficiently low.



(a) Prices (b) Returns (c) Market portfolio

(d) Ex-ante betas (e) Rolling estimates of betas

FIGURE 4.3. Illustration of the impact on the market of a change in δ at $t = 600$. (a) The equilibrium prices of the risky assets and value of the market portfolio; (b) the equilibrium returns of the risky assets and the market portfolio; (c) the market portfolio proportions; (d) the ex-ante betas of the risky assets; and (e) the rolling estimates of the betas. Here $\alpha = 0.15, n_f = 0.3, \lambda = 1.5, q = 0, K = 50$ and $\delta = 0.98$ for $t \leq t^* = 600$ and $\delta = 0.90$ for $t > t^* = 600$ (over $T = 1000$ time periods). The blue, green and red lines represent asset 1, 2, and 3, respectively.

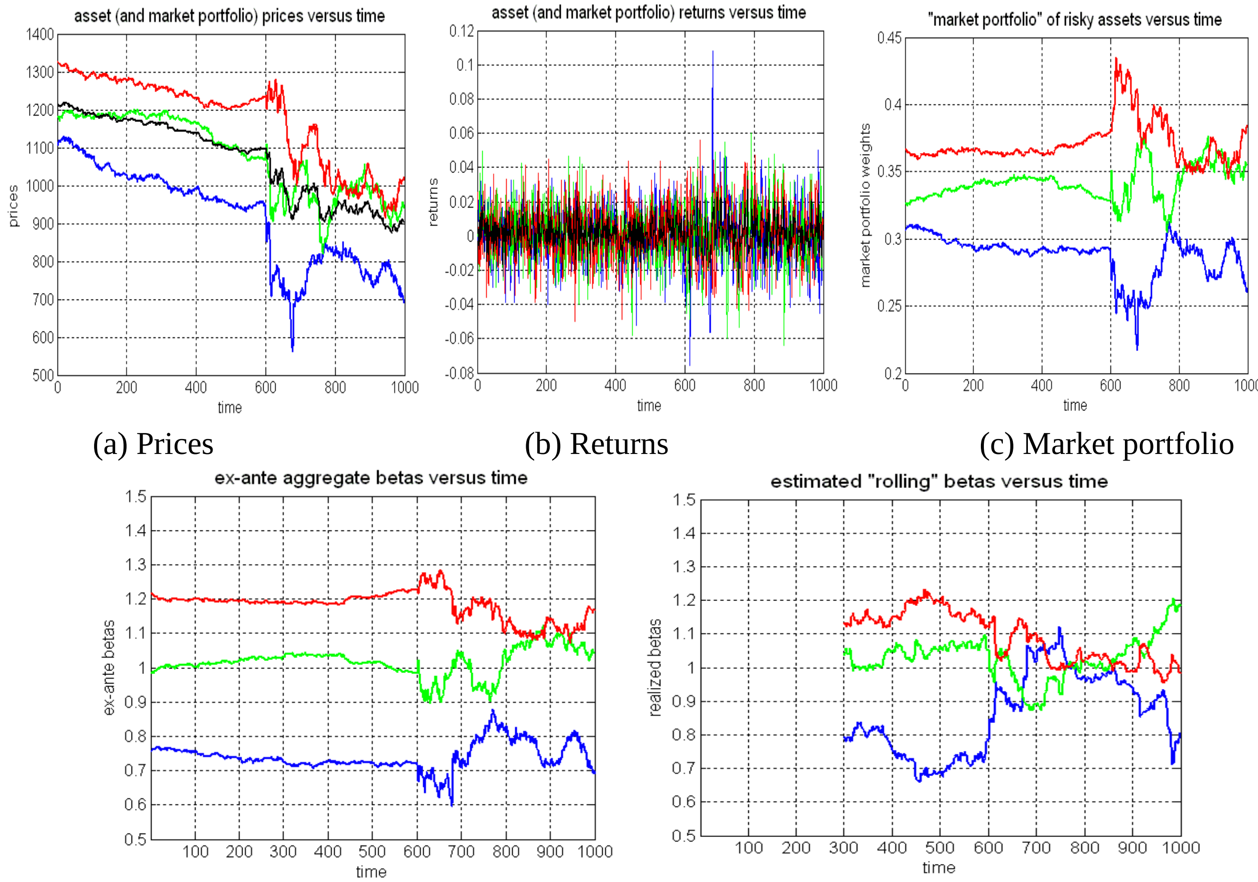
following examples we set agents' behavioral parameters as in section 4.1, namely, $\alpha = 0.15$, $\lambda = 1.5$, $n_f = 0.3$ and allow the decay rate δ to change to a different level at a certain time. For the parameter values, the fundamental traders expect a certain degree of mean reversion towards fundamental prices, whereas chartists update their beliefs about the expected returns and volatility/correlations based upon realized returns and observed deviations from sample average returns. Initially, we choose $\delta = \delta_1 = 0.98$, which is very close to 1. Then this scenario is still quite close to the benchmark case described above. We again adopt a weekly time step ($K = 50$), and the length of the simulation is $T = 1000$ time periods. A regime switch in δ occurs just after period $t^* = 600$ with δ decreasing from $\delta_1 = 0.98$ to $\delta_2 = 0.90$. The decrease in δ introduces a new phase with stronger trend extrapolation, that is, with chartists putting more weight on recent returns' history when forming their beliefs (3.5) and (3.6). For $\delta = 0.90$, the steady state equilibrium of the underlying deterministic model is still stable, though it is close to its deterministic bifurcation value[21], beyond which endogenous fluctuations emerge. By adding noise (from dividends or noise traders) into the model, dynamic patterns similar to the above mentioned deterministic fluctuations tend to emerge when the underlying deterministic steady state is still stable, provided that the parameter δ is close enough to the boundary of the region of stability (as is the case for the parameter value $\delta = \delta_2 = 0.90$). Therefore, this change in δ has a significant impact on the market equilibrium prices. Under the change, agents start varying their portfolios over time in order to exploit the emerging endogenous correlation patterns between the risky assets, sometimes reinforcing them. The impact is illustrated in Figs 4.3 and 4.4 resulting from two typical simulations with two different sample paths from the noisy dividend process.

Figs 4.3 and 4.4 indicate that in the first period with high $\delta = 0.98$, the equilibrium prices, returns, market weights and ex-ante aggregate betas fluctuate around their steady state levels, and that the dynamics in the initial period is not far from the reference case described in the stationary CAPM case. In this phase, the fluctuations are essentially driven by the exogenous noise. The parameter change then leads to a new scenario with more pronounced endogenous fluctuations of prices and returns. Such fluctuations also impact on the dynamics of market portfolio weights[22] and consequently on the time-varying ex-ante betas. In particular, we make the following observations.

Firstly, the stochastic nature of the time-varying betas changes significantly when the trend chasing behaviour of the trend followers changes. Due to the interaction of the extrapolation of the trend followers on the recent returns with mean-reverting activity from the fundamentalists, a strong extrapolation (measured by low δ) from the trend followers makes the market price fluctuate dramatically, as seen in Figs 4.3(a) and 4.4(a). This provides an opportunity for the trend followers to exploit emerging correlation among the risky assets, to re-balance their portfolios, which in turn affects the equilibrium prices and hence the composition of the market portfolio, as indicated in Figs 4.3(c) and 4.4(c). This expectation feedback mechanism leads to high volatility

[21]The value of $\delta = 0.90$ is near the bifurcation value for δ given by $\widehat{\delta}_A \in [0.879, 0.88]$. The previous deterministic analysis has demonstrated numerically the existence of a Neimark-Sacker bifurcation for decreasing values of the memory parameter δ (which implies increasing importance of recent market movements in trend follower beliefs).

[22]The relatively small amplitude of fluctuations of the price and return on the market portfolio contrasts with the large fluctuations of asset returns in the post-shock phase, which reveals that agents are actively trading in a way to exploit emerging correlations among the risky assets.

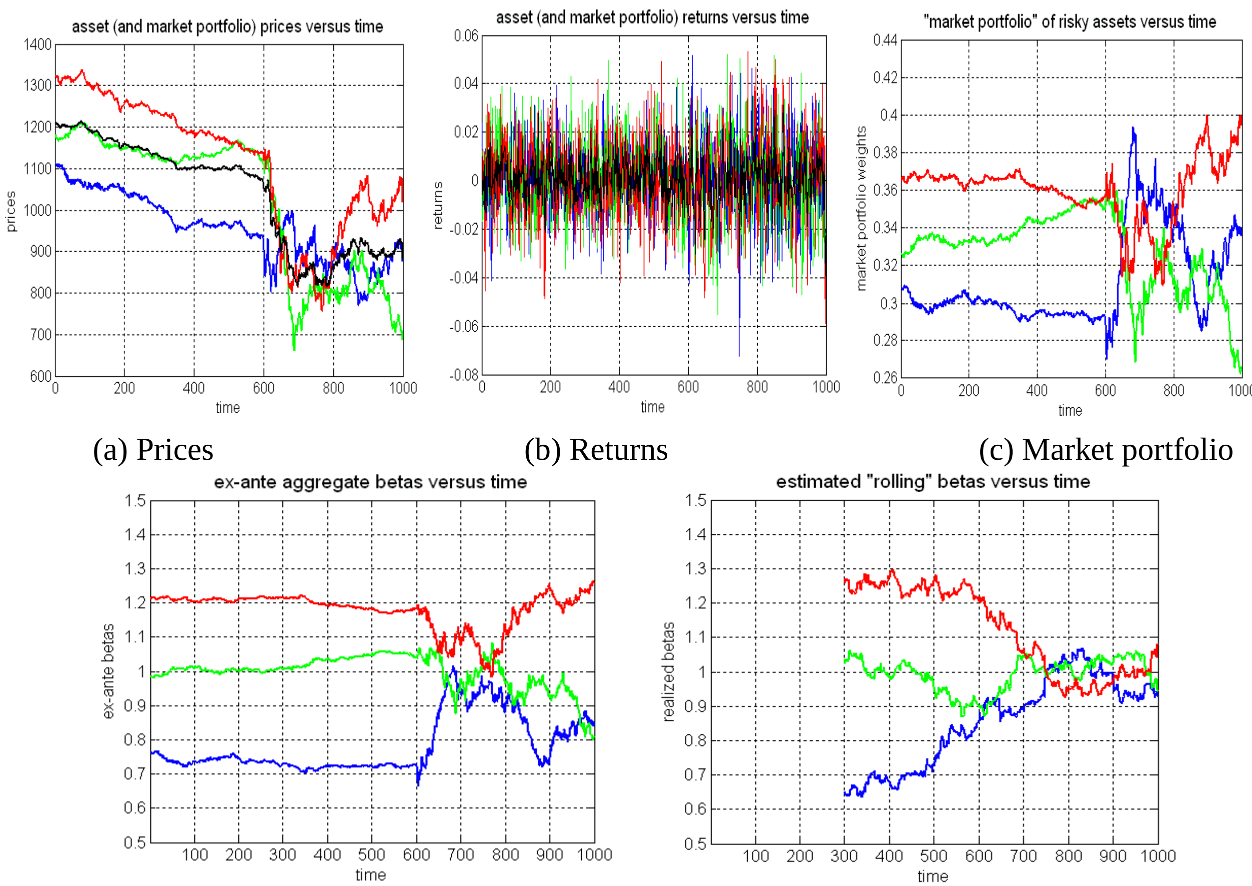


(a) Prices (b) Returns (c) Market portfolio

(d) Ex-ante betas (e) Rolling estimates of betas

FIGURE 4.4. Illustration of the impact on the market of a change in δ at $t = 600$. A different draw of the dividend process from Figures 4.3. (a) The equilibrium prices of the risky assets and value of the market portfolio; (b) the equilibrium returns of the risky assets and the market portfolio; (c) the market portfolio proportions; (d) the ex-ante betas of the risky assets; and (e) the rolling estimates of the betas. Here $\alpha = 0.15, n_f = 0.3, \lambda = 1.5, q = 0, K = 50$ and $\delta = 0.98$ for $t \leq t^* = 600$ and $\delta = 0.90$ for $t > t^* = 600$ (over $T = 1000$ time periods). The blue, green and red lines represent asset 1, 2, and 3, respectively.

in the market and the time-varying betas and simply reflects the change in risk of the risky assets, see Figs 4.3(d) and 4.4(d). In the period following the change, the average level of the time-varying betas of asset 3 is lower than the average ‘steady state’ betas in the initial period before the change, so that asset 3 becomes ‘less aggressive’. On the other hand, the average level of the time-varying betas of asset 1 is higher than the average ‘steady state’ betas in the initial period before the change, so that asset 1 becomes ‘less defensive’. In addition, measured by the time-varying ex-ante betas, asset 2, which is less risky than asset 3 before the change, can become more risky than asset 3 after the change, as illustrated in Figs 4.3(d) and 4.4(d). Even asset 1, which is much less aggressive than asset 3 before the change, can become almost as risky as asset 3 after the change (Figs 4.3(d)).



(a) Prices (b) Returns (c) Market portfolio

(d) Ex-ante betas (e) Rolling estimates of betas

FIGURE 4.5. Illustration of the impact on the market of a change in δ at $t = 600$. A different draw of the dividend process from Figures 4.3. (a) The equilibrium prices of the risky assets and value of the market portfolio; (b) the equilibrium returns of the risky assets and the market portfolio; (c) the market portfolio proportions; (d) the ex-ante betas of the risky assets; and (e) the rolling estimates of the betas. Here $\alpha = 0.15, n_f = 0.3, \lambda = 1.5, q = 0, K = 50$ and $\delta = 0.98$ for $t \leq t^* = 600$ and $\delta = 0.90$ for $t > t^* = 600$ (over $T = 1000$ time periods). The blue, green and red lines represent asset 1, 2, and 3, respectively.

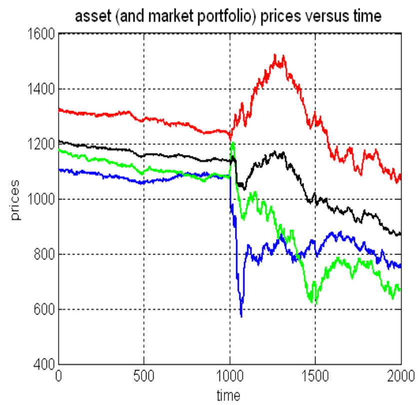
Secondly, Figs 4.3(e) and 4.4(e) suggest that the ex-post beta coefficients, estimated via rolling regression with a moving window of 300 periods, can display very different patterns from the ex-ante betas. In Fig. 4.3(e), the qualitative changes of rolling betas from the first to the second period seem to be somehow related to the changes occurring to the average levels of ex-ante betas. However, in Fig. 4.4(e), the pattern of the ex-post betas is not so similar to the ex-ante betas, although both ex-ante and ex-post rolling betas usually show important qualitative changes from one period to the other. In Fig. 4.4(e), the rolling estimates of time-varying betas for asset 1 after the change is very different from the ex-ante betas. After the change, the estimated betas of asset 1 vary between 0.7 and 1.1, while the variation of the ex-ante betas for the asset is much smaller and they remain largely below 1. This example indicates that rolling window estimates of time-varying betas can be misleading in an economy with boundedly rational agents. Finally, the example in Fig. 4.5, resulting from a different sample path of the dividend process (under the same parameter setting) again displays significant differences between the ex-ante betas and their rolling estimates in the last 200 periods of the time series.

Similar results can be obtained by assuming a monthly ($K = 12$) or a daily ($K = 250$) time step, and rescaling the parameters accordingly. Moreover, the results do not change substantially once *small* noise[23] is added to the supply (via the impact of noise traders). As an example, Fig. 4.6 reports the results of a simulation run with daily time steps ($K = 250$) and $T = 2000$ periods, where the standard deviation of the exogenous i.i.d. noise on the supply is equal to 0.15% of the average amount of shares. In this simulation, the chartist parameter δ is decreased from $\delta_1 = 0.98$ to $\delta_2 = 0.90$ at time $t^* = 1000$. The length of the rolling window is 500 periods. In this example, the inconsistency between the ex-ante betas and estimated betas becomes more significant and the estimated betas in the period after the shock vary between 0.6 and 1.2 for assets 1, between 0.9 and 1.25 for asset 3, in face of ex-ante betas that remain far more stable in the same period. Overall, we can see that such inconsistency is significant for assets 1 and 3, which are the least risky and the most risky assets, respectively. This observation may provide an explanation as to why in empirical studies, the time-varying CAPM based on the rolling window estimates of betas may have little or no explanatory power and this may simply due to the way the model is estimated rather than any shortcoming of the underlying equilibrium models.

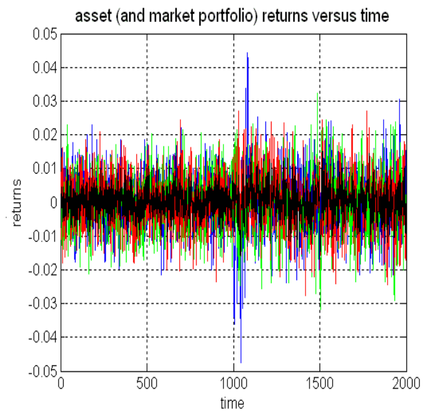
Similar experiments could be carried out by assuming that an exogenous shock at time t^* affects other behavioral parameter. A downward shift of parameters λ would result in scenarios somehow similar to those depicted above. The same would happen under a sudden increase of the impact of the noise-trader demand component (an upward shift of the parameter q). The next section contains a deeper analysis of the impact of changes of the key parameters.

We can summarise the insights provided by this experiment as follows. Roughly speaking, what matters when beliefs are approximately homogeneous and constant

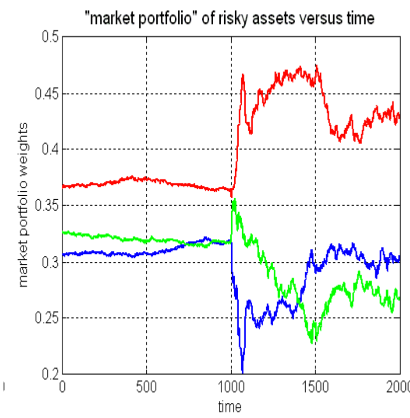
[23]Note that higher levels of 'market noise' may considerably affect the dynamics and determine large shifts of the average levels of the betas (see the numerical experiments in the next subsection). One reason is that in our simplified setup, unlike the dividend noise, market noise has no counterpart in the exogenous fixed portion of second-moment beliefs (that determines the steady state levels of the betas).



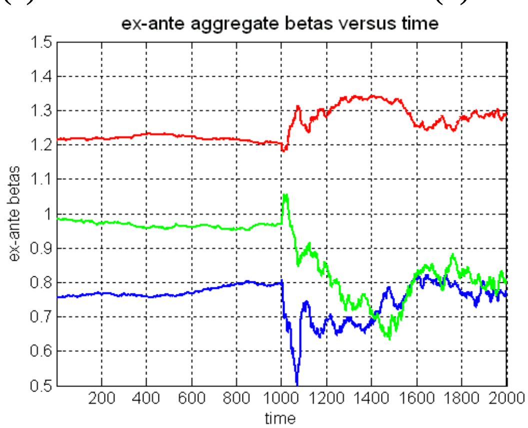
(a) Prices



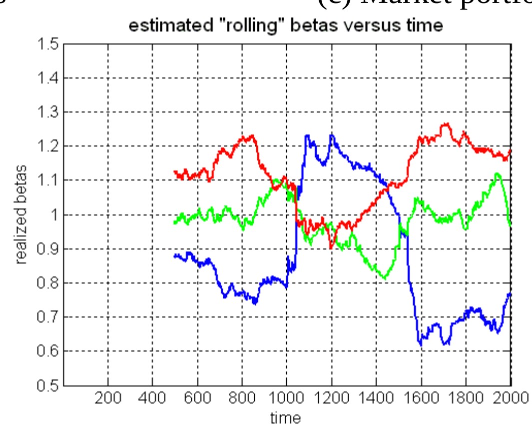
(b) Returns



(c) Market portfolio



(d) Ex-ante betas



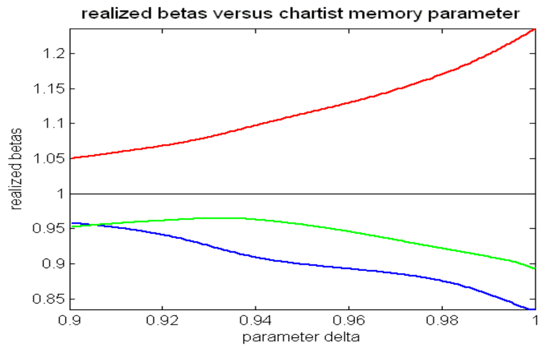
(e) Rolling estimates of betas

FIGURE 4.6. Illustration of the impact on the market of a change in δ at $t = 1000$ at daily frequency (a) The equilibrium prices of the risky assets and value of the market portfolio; (b) the equilibrium returns of the risky assets and the market portfolio; (c) the market portfolio proportions; (d) the ex-ante betas of the risky assets; and (e) the estimates of the betas using rolling window of 500. Here $\alpha = 0.15, n_f = 0.3, \lambda = 1.5, q = 0.15\%, K = 250$ and $\delta = 0.98$ for $t \leq t^* = 1000$ and $\delta = 0.90$ for $t > t^* = 1000$ (over $T = 2000$ time periods). The blue, green and red lines represent asset 1, 2, and 3, respectively.

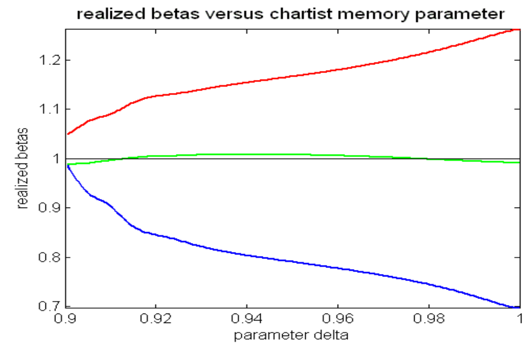
over time is the 'fundamental' part of agents beliefs about the expected returns and their variance/covariance matrix. In our example (without noise traders), such beliefs are consistent with the first and second moments of the resulting return process. As a consequence, when the steady state equilibrium of the underlying deterministic system is well within the region of stability, the estimated betas are consistent with ex-ante betas. However, when the steady state of the underlying deterministic system is destabilised via a particular bifurcation scenario, or is close to the stability boundary, stronger correlation patterns emerge from the noisy model, driven by time varying expectations and by the history-dependent portion of second-moment beliefs. Hence in equilibrium, the market portfolio also varies over time. This produces fluctuations of single-period ex-ante aggregate betas and, sometimes, a systematic change of their average level. Ex-post betas computed over different subperiods are affected by somehow similar changes. Since ex-ante betas are directly related to certain behavioural parameters, our findings indicate that the time variation of observed beta coefficients could be related, in principle, to changes in market sentiment. This interpretation has been suggested, so far, by 'visual inspection' of the time series of the 'rolling' betas. Note, however, that in the previous simulations the changes of the betas from one period to the other, affected by the parameter shift, could also be partly due to the way the underlying noise process develops over the two periods. Therefore, a more sound basis to such an interpretation is required. This will be provided in the next section, where we estimate the betas over a sufficiently long time window, for a full range of values of each parameter, under fixed sample paths for the exogenous noise processes.

4.4. **Parameter dependence of realized betas.** The numerical experiments in this section offer a deeper insight into the effect, on the beta coefficients, of the model behavioral parameters, namely, the chartist parameters δ (interpreted as the 'memory' of trend extrapolators' moving averages) and λ (sensitivity of risk beliefs to historical volatility/correlation), the fundamentalist mean reversion parameter α, and the parameter q, related to the standard deviation of noise trading. By assuming a time horizon of $T = 480$ iterations at monthly time step ($K = 12$) or, alternatively, a time horizon of $T = 1000$ iterations at weekly time step ($K = 50$) we simulate the noisy model for a grid of values of each parameter, within a specified range. Apart from the parameter that is allowed to vary, the remaining parameters are set according to our base selection $\alpha = 0.15$, $n_f = 0.3$, $\delta = 0.95$, $\lambda = 1.5$, $q = 0$ (that is, we assume absence of noise traders in the experiments involving the parameters α, n_f, δ and λ). For each simulation run, realised betas over the whole time horizon are plotted against different values of the the parameter under analysis, by assuming the same sample path for the dividend process. Fig. 4.7 displays the results of two different simulation runs for each parameter, under the monthly and weekly scenarios (left panels and right panels, respectively).

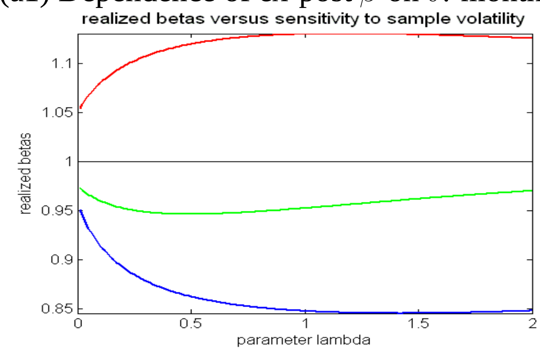
In Figs 4.7 (a1) and (a2), we vary the parameter δ in the range $[0.90, 1]$. While for δ close to 1, the realised betas are close to their steady state levels, systematic changes in betas can be observed as long as δ decreases. In particular, we note a tendency for the beta coefficients to become less dispersed as δ decreases. Intuitively, this may indicate that the more weight trend extrapolators put on recently observed price *changes* and *co-movements*, when forming their beliefs and portfolios, the more similar price and return patterns of the three assets become. As a consequence, the dynamic behavior of



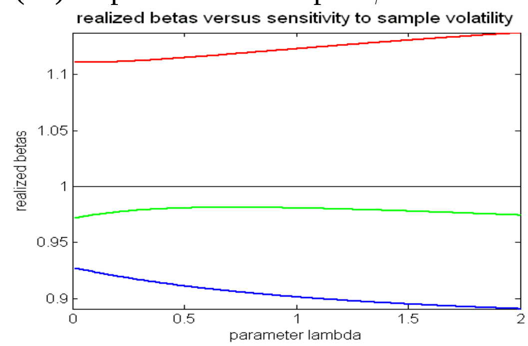
(a1) Dependence of ex-post β on δ: monthly



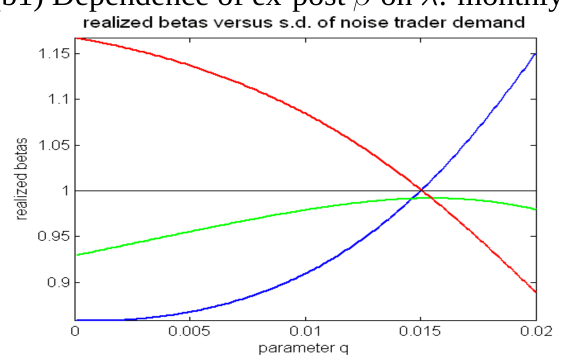
(a2) Dependence of ex-post β on δ: weekly



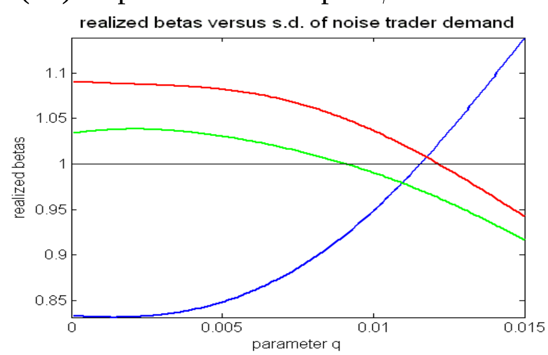
(b1) Dependence of ex-post β on λ: monthly



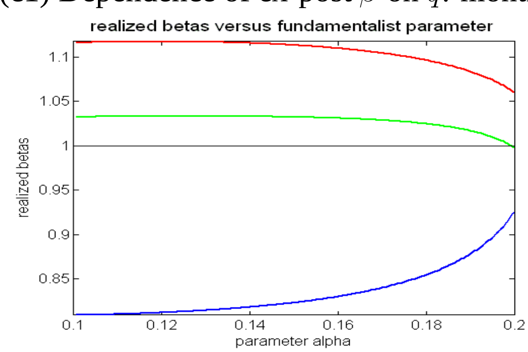
(b2) Dependence of ex-post β on λ: weekly



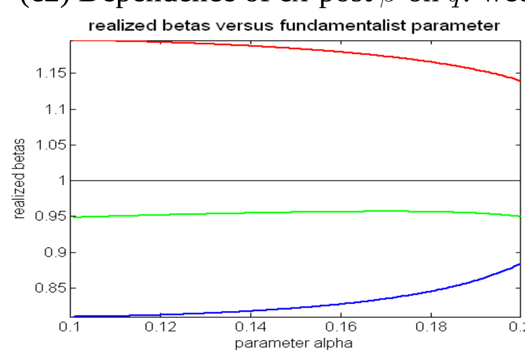
(c1) Dependence of ex-post β on q: monthly



(c2) Dependence of ex-post β on q: weekly



(d1) Dependence of ex-post β on α: monthly



(d2) Dependence of ex-post β on α: weekly

FIGURE 4.7. Dependence of realized betas on parameters δ, λ, q, α. Left panels: monthly time step, 480 iterations. Right panels: weekly time step, 1000 iterations. Base parameters: $\alpha = 0.15$, $\lambda = 1.5$, $n_f = 0.3$, $\delta = 0.95$, $q = 0$.

each asset tends to become increasingly similar to the market, in terms of stronger correlation and/or more similar level of volatility. A qualitatively similar effect (though less pronounced) is reported when sensitivity to sample variances/covariances (the parameter λ) decreases over the range $[0, 2]$, see Figs 4.7 (b1) and (b2). The effect of market noise (the parameter q) is also similar, when the level of noise is small. When the parameter q increases over the range $[0, 0.02]$ for monthly data, or in the range $[0, 0.015]$ for weekly data, see Figs 4.7 (c1) and (c2), the beta coefficients become less dispersed initially, whereas stronger noise may produce large shifts of the betas and reverse their ordering. Our results suggest that, for a given level of chartist memory parameter δ, smaller sensitivity to observed volatility (lower λ), and slightly larger market noise (higher q, within a suitable range), tend to strengthen the above described impact of parameter δ. The effect of varying parameter α is more ambiguous, in general. Under the assumed parameter selection, when α increases in the range $[0.1, 0.2]$ the betas tend to become less disperded, that is stronger mean reversion has a somehow similar effect to stronger trend extrapolation (see Figs 4.7 (d1) and (d2)). For different selection of the parameters, α may have the opposite effect. Similar effects are also observed by varying the parameter n_f (simulations are not reported here). From our numerical experiments it appears also that such phenomena are robust enough with respect to a different choice of the time steps.

Leaving aside the direction of the impact of parameter changes, and their possible interpretation, the main message from these experiments is that most of the key behavioral parameters are able to produce significant changes in both the ex-ante and the ex-post beta coefficients.

5. Conclusion

Although the conditional CAPM with time-varying betas display superiority in empirically explaining the cross-section of returns and anomalies, it is mostly motivated by econometric estimation and therefore lacks any economic foundations and intuition. In fact, financial market behaviour is the outcome of the interaction of investors who trade optimally for different purposes with different expectations. It is this heterogeneity and bounded rationality that has not been characterized in the current CAPM literature with time-varying betas. This paper aims to fill this missing part of the literature by modelling explicitly the time varying behaviour of the betas through agents' behaviour.

Motivated by the recent development of heterogeneous agent models, in this paper, we set up a boundedly rational dynamic equilibrium model of a financial market with heterogeneous agents within the mean-variance framework of repeated one-period optimisation. We first obtain an explicit dynamic CAPM relation between the expected equilibrium returns and time-varying betas. We then apply the results to a financial market model with heterogeneous agents by incorporating fundamentalists, trend followers and noise traders into the model. We show that, independently of the fundamentals, there is a systematic change in the market portfolio, asset prices and returns, and time varying betas when investors change their behaviour, captured by the trend extrapolation of the chartists, the mean-reversion of the fundamentalists, and the strength of the noise traders. In particular, we demonstrate the stochastic nature of time-varying betas and show that the commonly used rolling window estimates of time-varying betas are connected to, but may not be consistent with the ex-ante betas implied by the equilibrium model. The variation of the estimated betas can be significantly different from that of ex-ante betas. This observation may help us to understand some empirical findings that the time-varying CAPM based on the rolling window estimates of betas may have no explanatory power and this may simply be due to the estimation technique rather than some shortcoming of the underlying equilibrium models. The results provide some insights into the factors affecting the time variation of beta.

It would be interesting to examine the statistical properties of the asset returns, including the normality of the return distributions, volatility clustering, fat tails, and long memory in the asset returns, in particular, their dependence on agent behaviour. It would also be interesting to study the impact of adaptive behaviour when agents use combined strategies or beliefs in which the weights are updated by some fitness measure. We leave these issues to future research.

Appendix: Rescaling the chartist parameter δ to different trading periods

Chartist expected return (3.5) can be rewritten as a time average of past returns, with exponentially declining weights, as

$$\mathbf{u}_t = \delta \mathbf{u}_{t-1} + (1-\delta)\mathbf{r}_t = \sum_{s=0}^{\infty} \delta^s (1-\delta)\mathbf{r}_{t-s}. \tag{5.1}$$

Assume that time is measured in years. As is well known, the parameter δ, $0 < \delta < 1$, is linked to the average memory length

$$\ell := \sum_{s=0}^{\infty} s(1-\delta)\delta^s = (1-\delta)\sum_{s=0}^{\infty} s\delta^s = (1-\delta)\sum_{s=1}^{\infty} s\delta^s.$$

The latter is an arithmetic-geometric series, which sums up to

$$\sum_{s=1}^{\infty} s\delta^s = \frac{\delta}{(1-\delta)^2},$$

and therefore we obtain:

$$\ell = \frac{\delta}{1-\delta}.$$

This means that, when the chartists compute the sample time average using the weighting parameter δ, their average memory length is $\frac{\delta}{1-\delta}$ years.

If we switch, say, to monthly data, a reasonable criterion to rescale the chartist parameter to the new time step (denote the new parameter by $\delta^{(12)}$) is to keep the memory length constant, that is

$$\ell = \frac{\delta}{1-\delta} \text{ years } = \frac{\delta^{(12)}}{1-\delta^{(12)}} \text{ months.}$$

This means that δ and $\delta^{(12)}$ are related by

$$\frac{\delta^{(12)}}{1-\delta^{(12)}} = \frac{12\delta}{1-\delta}$$

from which

$$\delta^{(12)} = \frac{12\delta}{1+11\delta}.$$

In general, if K is the new time frequency and $\delta^{(K)}$ the rescaled decay parameter, then

$$\delta^{(K)} = \frac{K\delta}{1+(K-1)\delta},$$

where $0 < \delta^{(K)} < 1$. In this way, agents update time averages at monthly frequency using equation (5.1) with parameter $\delta^{(12)}$, and this is consistent with updating at yearly frequency with parameter δ.

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