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Intertemporal Investment Strategies under Inflation Risk

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Abstract

This paper studies intertemporal investment strategies under inflation risk by extending the intertemporal framework of Merton (1973) to include a stochastic price index. The stochastic price index gives rise to a two-tier evaluation system: agents maximize their utility of consumption in real terms while investment activities and wealth evolution are evaluated in nominal terms. We include inflation-indexed bonds in the agents' investment opportunity set and study their effectiveness in hedging against inflation risk. A new multi-factor term structure model is developed to price both inflation-indexed bonds and nominal bonds, and the optimal rules for intertemporal portfolio allocation, both with and without inflation-indexed bonds are obtained in closed form. The theoretical model is estimated using data of US bond yield, both real and nominal, and S&P 500 index. The estimation results are employed to construct the optimal investment strategy for an actual real market situation. Wachter (2003) pointed out that without inflation risk, the most risk averse agents (with an infinite risk aversion parameter) will invest all their wealth in the long term nominal bond maturing at the end of the investment horizon. We extend this result to the case with inflation risk and conclude that the most risk averse agents will now invest all their wealth in the inflation-indexed bond maturing at the end of the investment horizon.

Keywords: Inflation-Indexed Bonds; Intertemporal Asset Allocation; Inflationary Expectations

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1 Introduction

In a world with inflation risk a long-term bond is no longer a certain asset. Its payout at a future date is fixed but the purchasing power of the payout is unknown. For investors some important questions are; How to invest in nominal bonds in a world with inflation? Are long-term bonds still safer than short-term bonds? When there are *inflation-indexed bonds* (IIBs) on the market, what is the optimal portfolio containing the IIBs? This paper extends Merton's (1973) framework of intertemporal asset allocation to including a stochastic price index. The focuses of this paper are the study of the impacts of inflation risk and of the inclusion of IIBs on optimal investment strategies.

Inflation-indexed bonds are securities whose principal and coupon payments are adjusted with respect to some price index. They provide certain purchasing power and can hedge inflation risk for a long run investment plan. The US Treasury has been issuing Treasury Indexed-Protected Securities (TIPS) since January 1997, these are securities whose payments are adjusted to the Consumption Price Index. The outstanding amount of IIBs in 2004 was about $200bn in the US and $500bn worldwide. Liquidity in the TIPS market is improving, with the daily trading volume having doubled during 2002-2004 and amounting to about $5bn in 2004.[1]

Although there have been many contributions to the problem of intertemporal asset allocation since the pioneering work of Merton, such as Kim and Omberg (1996), Brennan, Schwartz and Lagnado (1997), Wachter (2002, 2003), Liu(2005) and others, models considering inflation risk are still in the developmental stage. Campbell and Viceira (2001) solve the intertemporal asset allocation problem of infinitely-lived agents with recursive utility under inflation risk. The no-arbitrage constraint of their discrete-time model is represented by a pricing formula in terms of a *real* stochastic discount factor (SDF). The continuous-time asset model provided by Brennan and Xia (2002) suggests an analogous pricing scheme that uses a *real pricing kernel*. They

[1] Details see the source: http://www.treas.gov/offices/domestic-finance/key-initiatives/tips.shtml

provide a solution solve for the optimal portfolio consisting of investment in one equity and two nominal bonds but without IIBs. The application of the real pricing kernel still remains many unexplored issues. The theoretical foundation for the pricing kernel is its equivalence to the no-arbitrage constraint that is guaranteed by frictionless and efficient transactions on markets. Since all transactions on markets take place in *nominal* terms, the no-arbitrage constraint should only be equivalent to a pricing kernel in *nominal* terms. A pricing kernel in real terms seems to be unconvincing because it requires frictionless and efficient transactions in units of goods.

In a world with inflation, it is more convincing to adopt the no-arbitrage constraint developed by Jarrow and Yildirim (2003). They consider the "nominal world" and the "real world" as two countries and the price index as the "exchange rate" based on the no-arbitrage constraint for the two-country model proposed by Amin and Jarrow (1991). Invoking an argument analogous to that in the two-country model that the no-arbitrage constraint is satisfied on each national financial market, Jarrow and Yildirim obtain the no-arbitrage constraint for the "nominal world". However, we do not adopt their model directly here because their nominal term structure is based on a one-factor model, as in Munk et al. (2004). The shortcoming of such a one-factor model is that usually the factor is the instantaneous nominal spot interest rate. It then turns out that the inflation risk does not affect the nominal term structure. Furthermore, it is well known that a one-factor bond model does not fit market data well. We would thus expect to encounter difficulties in empirical applications of portfolio allocation rules based on single-factor models.

The model we develop adopts the no-arbitrage condition of Jarrow and Yildirim (2003) but we extend the one-factor nominal bond model framework to that of a two-factor model of the type proposed by Richard (1978) where both the instantaneous real interest rate and the instantaneous expected inflation rate are factors for the nominal term structure.

There are mainly two approaches for solving the intertemporal decision problem under inflation risk: the method of dynamic programming and the static variational method. The latter approach was employed by Brennan and Xia (2002) which was developed for their bond pricing model using the real pricing kernel. Since we have already argued above for a desire of more deeper theoretical analysis for the application of the real pricing kernel, it is technically more clear and direct to adopt the approach of dynamic programming. Munk et al. (2004) have provided solutions using the dynamic programming approach for the intertemporal investment problem under inflation risk. However, in their paper only the end solution is provided. We amend some of the solution steps and extend the underlying bond pricing model to including inflation expectations as one additional factor.

By including a stock with a constant risk premium, we end up considering an intertemporal model whose investment opportunity set includes a stock, nominal bonds and IIBs. We investigate optimal investment strategies in this framework. In this paper, we make use of the Feymann-Kac Formula to obtain the solution of the intertemporal portfolio choice problem, both with and without the IIBs, in closed form.

In a world without inflation risk, Wachter (2003) has shown that the most risk averse agents would only buy the nominal bond maturing at the end of the investment horizon. The reasoning for this investment strategy is that the nominal bond provides a certain payout at the end of the investment horizon. On a world *with* inflation risk, however, the nominal bonds are no longer safe assets. We provide the result that, under inflation risk, the most risk averse agents now invest all their wealth in the IIBs maturing at the end of investment horizon if the IIB is included in the investment opportunity set. This finding shares some sharing similar essence with that of Wachter (2003). If there is no IIB in the investment opportunity set, investors can hedge inflation risk only through the correlations between the asset return shocks and inflation shocks. The most risk averse investors still prefer to invest in the long-term bond.

The structure of this paper is organized as follows. Section 2 introduces the assets available for investment. The novel feature here is the embedding of the two-factor nominal bond model of Richard (1978) into the arbitrage model of Jarrow and Yildirim (2003). Section 3 solves the intertemporal investment problem for the world with inflation risk by using the *Feymann-Kac formula*. Any useful portfolio recommendation should be based on information reflected by markets. Section 4 investigates current markets empirically and provides the required information for the construction of the optimal intertemporal investment strategies. In Section 5, both optimal intertemporal investment strategies with and without IIBs are provided based real market situations. Section 6 draws conclusions and suggests future research directions. The proofs of various technical results are gathered in the appendices.

The extension of Merton's continuous-time framework for asset allocation in this paper is carried out by considering a time-varying price index I_t modelled by the diffusion process

$$\frac{dI_t}{I_t} = \pi_t dt + \sigma_I dW_t^I \ , \tag{1}$$

where W_t^I is a Wiener process and π_t is the *anticipated* instantaneous inflation rate [2]. A price index represents the price for a fixed basket of goods. The time-varying price index in our model gives rise to two evaluation terms: the nominal terms value in terms of money and the real terms value in terms of goods.

[2]See Richard (1978) .

2 A Multi-Factor Model for Nominal and Inflation-Indexed Bonds

Let $P_n(t,T)$ denote the zero-coupon **nominal bond** at t with maturity date T. The payout of the nominal bond is normalized as one money unit so that

$$P_n(T,T) = 1 \,. \tag{2}$$

Following Richard (1978) we assume that the instantaneous real interest rate r_t and the anticipated instantaneous inflation rate π_t are the two factors driving the nominal bond price. The two factors are assumed to follow the Gaussian mean-reverting processes

$$dr_t = \kappa_r(\overline{r} - r_t)dt + g_r dW_t^r \,, \tag{3}$$

and

$$d\pi_t = \kappa_\pi(\overline{\pi} - \pi_t)dt + g_\pi dW_t^\pi \,. \tag{4}$$

where W_t^r and W_t^π are correlated Wiener processes with the instantaneous variance $dW_t^r dW_t^\pi = \rho_{r\pi} dt$.

In this framework, the bond pricing formula belongs to the *exponential affine family* (the Duffie-Kan family)

$$P_n(r_t, \pi_t, t, T) = \exp\big(- A_n(T-t) - B_{nr}(T-t)r_t - B_{n\pi}(T-t)\pi_t \big) \,, \tag{5}$$

where the Duffie-Kan coefficients $A_n(\tau)$, $B_{nr}(\tau)$ and $B_{n\pi}(\tau)$ will be determined later by the no-arbitrage condition (23). The coefficients, due to the normalization (2), have the terminal conditions at maturity date given by

$$A_n(0) = 0 \,, \quad B_{nr}(0) = 0 \,, \quad B_{n\pi}(0) = 0 \,. \tag{6}$$

Applying Itô Lemma to (5), we can write the return of the nominal bond as

$$\frac{dP_n(t,T,r_t,\pi_t)}{P_n(t,T,r_t,\pi_t)} = \mu_n(t,T-t)dt - B_{nr}(T-t)g_r dW_t^r - B_{n\pi}(T-t)g_\pi dW_t^\pi \,, \tag{7}$$

where (setting $\tau = T - t$)

$$\begin{aligned} \mu_n(t,\tau) \quad := \quad & \frac{d}{d\tau}A_n(\tau) + \frac{d}{d\tau}B_{nr}(\tau)r_t + \frac{d}{d\tau}B_{n\pi}(\tau)\pi_t \\ & -B_{nr}(\tau)\kappa_r(\overline{r} - r_t) - B_{n\pi}(\tau)\kappa_\pi(\overline{\pi} - \pi_t) \\ & +\frac{1}{2}\left(B_{nr}(\tau)^2 g_r^2 + 2B_{nr}(\tau)B_{n\pi}(\tau)\sigma_r\sigma_\pi\rho_{r\pi} + B_{n\pi}(\tau)^2 g_\pi^2\right) . \end{aligned} \tag{8}$$

The nominal yield is defined by[3]

$$Y_n(t,T) := \frac{-\ln P_n(t,T)}{T-t} = \frac{A_n(T-t)}{T-t} + \frac{B_{nr}(T-t)}{T-t}r_t + \frac{B_{n\pi}(T-t)}{T-t}\pi_t \ . \tag{9}$$

The instantaneous nominal interest rate R_t is defined as the instantaneous yield, given by

$$R_t := \lim_{T \downarrow t} Y_n(t,T) \ . \tag{10}$$

Applying this last result to the yield formula (9), we then have the expression

$$R_t = A_n'(0) + B_{nr}'(0)r_t + B_{n\pi}'(0)\pi_t \ , \tag{11}$$

where A' denotes the derivative of A. The nominal money account is defined as the accumulation account

$$M_n(t) = \exp(\int_0^t R_s ds) \ . \tag{12}$$

Let $P_I(t,T)$ denote the price of the (zero-coupon) **inflation-indexed bond** (IIB) that is issued at time 0[4] and matures at time T. The payout at the maturity date will be adjusted by the price index I_T, therefore

$$P_I(T,T) = I_T. \tag{13}$$

Define the *real bond* $P_r(t,T) := P_I(t,T)/I_t$ as the normalized IIB with respect to the corresponding price index. According to (13), we have $P_r(T,T) = 1$, which means that the real bond has a payout of one unit of consumption good at T. We assume that the real bond is

[3]Strictly speaking all bond prices and yields should have as arguments r_t, π_t, t, T. However when the contex is clear we shall suppress the arguments r_t, π_t and mainly focus on the t, T dependence.

[4]We fix $I_0 = 1$

only affected by one factor, the instantaneous real interest rate r_t which also follows the Duffie and Kan dynamics

$$P_r(t,T) = \exp\big(- A_r(T-t) - B_{rr}(T-t)r_t\big) \ , \tag{14}$$

where the Duffie-Kan coefficients $A_r(\tau)$ and $B_{rr}(\tau)$ will be determined later by the no-arbitrage conditions (24) and (25).

The assumption (14) concerning the real bond implies the dynamics for $P_I(t,T)$ that will be showed below. The terminal condition (13) implies the conditions

$$A_r(0) = 0 \ , \qquad B_{rr}(0) = 0 \ . \tag{15}$$

The real yield is the constant interest rate of the real bond which is defined as

$$Y_r(t,T) := \frac{-\ln P_r(t,T)}{T-t} = \frac{A_r(T-t)}{T-t} + \frac{B_{rr}(T-t)}{T-t} r_t \ . \tag{16}$$

We denote a *consumption good account* $M_r(t)$ as

$$M_r(t) := \exp(\int_0^t r_s ds) \ ,$$

and $M_I(t)$ as the *real money account*, which gives the nominal value of the consumption good account and is expressed as

$$M_I(t) := M_r(t) I_t \ . \tag{17}$$

To calculate return of the IIB, we apply Itô's Lemma at first to the real bond price (14) and obtain

$$\frac{dP_r(t,T,r_t)}{P_r(t,T,r_t)} = \mu_r(t,T-t)dt - B_{rr}(T-t)g_r dW_t^r \ , \tag{18}$$

where

$$\mu_r(t,\tau) = \frac{d}{d\tau}A_r(\tau) + \frac{d}{d\tau}B_{rr}(\tau)r_t - B_{rr}(\tau)\kappa_r(\bar{r} - r_t) + \frac{1}{2}B_{rr}(T-t)^2 g_r^2 \ . \tag{19}$$

Next applying Itô's Lemma to the expression for the IIB,

$$P_I(t,T,r_t,I_t) = P_r(t,T,r_t)I_t \ ,$$

and recalling that the price index I_t follows the dynamics (1), we then obtain the return process of the IIB, namely

$$\frac{dP_I(t,T,r_t,I_t)}{P_I(t,T,r_t,I_t)} = \mu_I(t,T-t)dt - B_{rr}(T-t)g_r dW_t^r + \sigma_I dW_t^I \ , \tag{20}$$

where

$$\mu_I(t,T-t) := \mu_r(t,T-t) + \pi_t - B_{rr}(T-t)g_r\sigma_I\rho_{Ir} \ , \tag{21}$$

with $\rho_{Ir}dt := dW_t^r dW_t^I$.

The return on the real money account $M_I(t)$ can be calculated easily from (17) to be

$$\frac{dM_I(t)}{M_I(t)} = (r_t + \pi_t)dt + \sigma_I dW_t^I \ . \tag{22}$$

In order to obtain the bond price, we employ the standard *no-arbitrage* argument, see, for example, Chiarella (2004)[5]. It requires that the excess return should be equal to risk premia for the nominal bonds, the IIB and the real money account, so that we have the conditions

$$\mu_n(t,\tau) - R_t = -B_{nr}(\tau)g_r\lambda_r - B_{n\pi}(\tau)g_\pi\lambda_\pi \ , \quad \forall \tau > 0 \tag{23}$$

$$\mu_I(t,\tau) - R_t = -B_{rr}(\tau)g_r\lambda_r + \lambda_I\sigma_I \ , \quad \forall \tau > 0 \tag{24}$$

$$\pi_t + r_t - R_t = \lambda_I\sigma_I \ , \tag{25}$$

where $\mu_n(t,\tau), \mu_I(t,\tau)$ as defined in equations (8) and (21) and λ_r, λ_π, and λ_I are constants, usually interpreted as the market prices of risk associated respectively with the sources of risk W_t^r, W_t^π and W_t^I.

We make two remarks concerning the no-arbitrage conditions (23) – (25). First, this system of the no-arbitrage conditions (23) – (25) satisfies the no-arbitrage requirement in Jarrow

[5] Chapter 24. Interest Rate Derivatives-Spot Rate Models.

and Yildirim (2003)) (see their equation (8)). However, our nominal bond model (5) has two factors and is different from the single-factor model in Jarrow and Yildirim (2003)[6]. Comparing our conditions with theirs, the conditions given in (24) and (25) are equivalent to the conditions given in their equations (10b) and (10c). Their condition (10a) in the HJM setting is equivalent to the spot rate setting, see, Chiarella (2004)[7]

$$\mu_n(t,\tau) - R_t = -B_{nn}(\tau) g_r \lambda_n \ .$$

From this condition, we can see clearly their single-factor structure for the nominal bond pricing formula. Second, the usual signs (for positive excess return) for the market prices of risk are given by $\lambda_r < 0$, $\lambda_\pi < 0$ and $\lambda_I > 0$. Later we shall check for these in our empirical results.

Property 1 *If the no-arbitrage equalities (23) – (25) are satisfied, then*

(i) the coefficients $A_n(\tau), B_{nr}(\tau), B_{n\pi}(\tau)$ for the nominal bond price (5) are solved as

$$B_{nr}(\tau) = \frac{1}{\kappa_r}\big(1 - e^{-\kappa_r \tau}\big) \ , \tag{26}$$

$$B_{n\pi}(\tau) = \frac{1}{\kappa_\pi}\big(1 - e^{-\kappa_\pi \tau}\big) \ , \tag{27}$$

$$\begin{aligned} \frac{A_n(\tau)}{\tau} = {} & \big(1 - \frac{1}{\tau\kappa_r} + \frac{e^{-\tau\kappa_r}}{\tau\kappa_r}\big)\big(\bar{r} - \frac{g_r\lambda_r}{\kappa_r}\big) + \big(1 - \frac{1}{\tau\kappa_\pi} + \frac{e^{-\tau\kappa_\pi}}{\tau\kappa_\pi}\big)\big(\bar{\pi} - \frac{g_\pi\lambda_\pi}{\kappa_\pi}\big) \\ & - \frac{g_r^2}{2\kappa_r^2}\big(1 - 2\frac{1-e^{-\kappa_r\tau}}{\kappa_r\tau} + \frac{1-e^{-2\kappa_r\tau}}{2\kappa_r\tau}\big) - \frac{g_\pi^2}{2\kappa_\pi^2}\big(1 - 2\frac{1-e^{-\kappa_\pi\tau}}{\kappa_\pi\tau} + \frac{1-e^{-2\kappa_\pi\tau}}{2\kappa_\pi\tau}\big) \\ & - \frac{g_r g_\pi \rho_{r\pi}}{\kappa_r\kappa_\pi}\big(1 - \frac{1-e^{-\kappa_r\tau}}{\kappa_r\tau} - \frac{1-e^{-\kappa_\pi\tau}}{\kappa_\pi\tau} + \frac{1-e^{-(\kappa_r+\kappa_\pi)\tau}}{(\kappa_r+\kappa_\pi)\tau}\big) + \xi_0 \ . \end{aligned} \tag{28}$$

(ii) The coefficients $A_r(\tau), B_r(\tau)$ for the real yield (14) are solved as

$$B_{rr}(\tau) = \frac{1}{\kappa_r}\big(1 - e^{-\kappa_r \tau}\big) \tag{29}$$

$$\begin{aligned} \frac{A_r(\tau)}{\tau} = {} & \big(1 - \frac{1}{\tau\kappa_r} + \frac{e^{-\tau\kappa_r}}{\tau\kappa_r}\big)\big(\bar{r} - g_r\frac{\lambda_r - \sigma_I\rho_{Ir}}{\kappa_r}\big) \\ & - \frac{g_r^2}{2\kappa_r^2}\big(1 - 2\frac{1-e^{-\kappa_r\tau}}{\kappa_r\tau} + \frac{1-e^{-2\kappa_r\tau}}{2\kappa_r\tau}\big) \ . \end{aligned} \tag{30}$$

[6] The single-factor setting can be seen in the part "B. Volatility Parameters for the Nominal Forward Rates" on p.351 in their paper.

[7] Chapter 25. The Heath-Jarrow-Morton Model

Property 2 *If the no-arbitrage equalities (23) – (25) are satisfied, then (i) the instantaneous nominal interest rate is given by*

$$R_t = \xi_0 + r_t + \pi_t \ . \tag{31}$$

(ii) When the IIBs are included in the investment set, then we have

$$\xi_0 = -\lambda_I \sigma_I \ . \tag{32}$$

3 Intertemporal Investment Strategies with Inflation Risk

3.1 The Investment Opportunity Set

The investment opportunity set includes five assets: the nominal money account, two nominal bonds with different maturities T_1, T_2, one IIB maturing at T_3 and one stock. The stock price is assumed to follow the geometric Wiener process

$$\frac{dP_S(t)}{P_S(t)} = (R_t + \lambda_S \sigma_S)dt + \sigma_S dW_t^S \ , \tag{33}$$

with $\sigma_S > 0$ a constant instantaneous standard deviation of stock returns and $\lambda_S > 0$ a constant market price of risk associated with the uncertainty W_t^S.

Summarizing all risky asset returns according to (7), (20) and (33) in vector form, we write

$$\begin{pmatrix} dP_n(t,T_1)/P_n(t,T_1) \\ dP_n(t,T_2)/P_n(t,T_2) \\ dP_I(t,T_3)/P_I(t,T_3) \\ dP_S(t)/P_S(t) \end{pmatrix} = \mu_t dt + \Sigma_t dW_t \tag{34}$$

where

$$\mu_t = R_t\underline{\mathbf{1}} + \Sigma_t\lambda\,, \tag{35}$$

$$\Sigma_t := \begin{pmatrix} -B_{nr}(T_1-t)g_r & -B_{n\pi}(T_1-t)g_\pi & 0 & 0 \\ -B_{nr}(T_2-t)g_r & -B_{n\pi}(T_2-t)g_\pi & 0 & 0 \\ -B_{rr}(T-t)g_r & 0 & \sigma_I & 0 \\ 0 & 0 & 0 & \sigma_S \end{pmatrix}, \tag{36}$$

$$\underline{\mathbf{1}} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad dW_t := \begin{pmatrix} dW_t^r \\ dW_t^\pi \\ dW_t^I \\ dW_t^S \end{pmatrix}, \quad \text{and} \quad \lambda := \begin{pmatrix} \lambda_r \\ \lambda_\pi \\ \lambda_I \\ \lambda_S \end{pmatrix}. \tag{37}$$

The equality (35) is because of the no-arbitrage conditions (23), (24) and the stock return dynamics (33).

The four risks factors $dW_t^r, dW_t^\pi, dW_t^I, dW_t^S$ are correlated with the correlation matrix $\mathcal{R}_{AA}dt := dW_t dW_t^\top$. The correlation matrix between W_t and W_t^I is denoted by $\mathcal{R}_{AI}dt = dW_t dW_t^I$.

3.2 The Model

Adopting Merton's setting, we assume that there are identical agents who are endowed with V_0 units of wealth (nominal value) at time 0 and seek to maximize their expected final utility at T,

$$\max_{\alpha_t, t\in[0,T]} \mathbf{E}_0\big[U(v_T)\big]\,. \tag{38}$$

The lower case v_t represents the *real* wealth which is by definition given by $v_t := V_t/I_t$. The utility is of the constant relative risk aversion (CRRA) class,

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}\,, \tag{39}$$

where $\gamma > 0$ represents the relative risk aversion (RRA) coefficient. The agents can revise their investment decision α_t without transaction costs for any time $t \in [0, T]$ where $\alpha_t := \big(\alpha_{it}\big)_{i=1,\cdots,4}$ and each α_{it} represents the investment proportion in the i-th risky asset. The investment amount has to be balanced by the nominal money account $M_n(t)$ so its proportion α_{0t} turns out to be equal to $\alpha_{0t} = 1 - \sum_{i=1}^{4} \alpha_{it}$.

Given the decisions α_t, agents' wealth evolves following the dynamics

$$\frac{dV_t}{V_t} = \sum_{i=0}^{4} \alpha_{it} \frac{dP_{it}}{P_{it}} = R_t dt + \alpha_t^\top \Big((\mu_t - R_t \underline{\mathbf{1}}) dt + \Sigma_t dW_t \Big) , \tag{40}$$

where $\alpha_t^\top = (\alpha_{1t}, \cdots, \alpha_{4t})$, μ_t is the expected return vector and Σ_t is the volatility matrix defined in (36).

To obtain the evolution of the real wealth $v_t = V_t/I_t$, at first we apply Itô's Lemma to the inverse of the price index process (1) and obtain

$$d(\frac{1}{I_t}) = \frac{1}{I_t}\big(- \pi_t dt + \sigma_I^2 dt - \sigma_I dW_t^I \big) . \tag{41}$$

Applying Itô's Lemma again to $v_t = V_t/I_t$ and using the result of the nominal wealth evolution (40), we obtain the evolution of the real wealth dynamics,

$$\begin{aligned} \frac{dv_t}{v_t} &= \big(R_t - \pi_t + \sigma_I^2\big) dt + \alpha_t^\top \big(\mu_t - R_t \underline{\mathbf{1}} - \sigma_I \Sigma_t \mathcal{R}_I\big) dt \\ &\quad + \alpha^\top \Sigma_t dW_t - \sigma_I dW_t^I . \end{aligned} \tag{42}$$

Now, the agents' investment decision problem is to find the optimal path α_t for $t \in [0, T]$, which maximizes the objective function (38) under the *real* budget constraint (42) and the factor dynamics (3) and (4).

3.3 Solving via the method of dynamic programming

As mentioned in the introduction, we employ *dynamic programming*, as proposed by Merton (1971), to solve the intertemporal decision problem in Section 3.2.

The underlying factors affecting the asset return dynamics in this model are the instantaneous real interest rate r_t and the instantaneous expected inflation rate π_t. We use X_t to denote these factors so that $X_t = \big(r_t, \pi_t\big)^\top$. Summarizing the factor dynamics (3) and (4) in vector form we write

$$dX_t = F_t dt + G_t dW_t^X \ , \tag{43}$$

where the functions F and G are defined by

$$F_t := \begin{pmatrix} \kappa_r(\overline{r} - r_t) \\ \\ \kappa_\pi(\overline{\pi} - \pi_t) \end{pmatrix} \ , \quad G_t := \begin{pmatrix} g_r & 0 \\ \\ 0 & g_\pi \end{pmatrix} \ . \tag{44}$$

Also, we have $W_t^X = \big(W_t^r, W_t^\pi\big)^\top$ and the correlation matrix of which is denoted by $\mathcal{R}_{XX} dt := dW_t^X dW_t^{X\top}$.

Let $J(t, T, v_t, X_t)$ denote value function (the optimized objective function) over a subperiod $[t, T]$ with given initial real wealth v_t and the given state of the factor X_t, that is[8]

$$J(t, T, v_t, X_t) := e^{-\delta T} \max_{\alpha_s, s \in [t,T]} \mathbf{E}_t\big[U(v_T)\big] \ . \tag{45}$$

[8] The definition of the value function $J(t, T, v_t, X_t)$ is different from that given in (38). However, the discount factor $e^{-\delta T}$ in equation (45) is only a constant so it does not affect the optimal path of the portfolio decision $\alpha_s, s \in [t, T]$.

The key result of the dynamic programming approach is that the value function has to satisfy the *Hamilton-Jacobi-Bellman*(HJB) equation[9]

$$
\begin{aligned}
0 = \max_{\alpha_t} \quad \Big\{ \quad & \big(R_t - \pi_t + \sigma_I^2 + \alpha_t^\top(\mu_t - R_t\underline{\mathbf{1}} - \Sigma_t \mathcal{R}_{AI}\sigma_I)\,\big) J_v v_t \\
& + \frac{1}{2}\big(\alpha_t^\top \Sigma_t \mathcal{R}_{AA} \Sigma_t^\top \alpha_t - 2\sigma_I \alpha_t^\top \Sigma_t \mathcal{R}_{AI} + \sigma_I^2\big) J_{vv} v_t^2 \\
& + \big(\alpha_t^\top \Sigma_t \mathcal{R}_{AX} G_t^\top - \sigma_I \mathcal{R}_{IX} G_t^\top\big) J_{vX}\, v_t \\
& + F_t^\top J_X + \frac{1}{2}\sum_{i,j=1}^{2} G_{it}\mathcal{R}_{XX} G_{jt}^\top J_{X_i X_j} + \frac{\partial}{\partial t} J \Big\} ,
\end{aligned} \tag{47}
$$

where $\mathcal{R}_{XA}dt := dW_t^X dW_t^\top$, $\mathcal{R}_{AI}dt := dW_t dW_t^I$, $\mathcal{R}_{XI}dt := dW_t^X dW_t^I$ and G_{it} denotes the i-th row of the matrix G_t. The J written with subscript represents the relevant partial derivative.

We observe that the optimal portfolio $\alpha_s, s \in [t,T]$ is independent of the initial wealth level v_t because the CRRA utility function is homothetic[10], and the dynamics $\frac{dv_s}{v_s}$ and dX_s are independent of v_t. We note that

$$
J(t,T,v_t,X_t) = v_t^{1-\gamma} e^{-\delta T} \max_{\alpha_s; t \le s \le T} \Big\{ \mathbf{E}_t[U(\frac{v_T}{v_t})] \Big\} = v_t^{1-\gamma} J(t,T,1,X_t) \,,
$$

and so we can decompose $J(t,T,v_t,X_t)$ into

$$
J(t,T,v_t,X_t) = e^{-\delta t} U(v_t) \Phi(t,T,X_t)^\gamma \,, \tag{48}
$$

where

$$
\Phi(t,T,X_t)^\gamma := e^{\delta t}(1-\gamma) J(t,T,1,X_t) \,. \tag{49}
$$

[9]The intuition behind the HJB equation lies in the infinitesimal decomposition

$$
J(t,T,v_t,X_t) = \max_{\alpha_t} \big\{ J(t+dt, T, v_{t+dt}, X_{t+dt}) \big\} \,. \tag{46}
$$

See P.264-271 in Kamien and Schwartz (1991) for a heuristic discussion and Chapter 11 in Øksendal(2000) for a rigorous derivation. Note that the intermediate consumption is not considered in the agents' objective function (38) therefore the intermediate consumption does not appear in the infinitesimal decomposition (46). The HJB equation represents a necessary condition for the value function.

[10]A function is homothetic if it can be decomposed into an inner function that is monotonically increasing and an outer function that is homogeneous of degree one.

Applying the first order condition for α_t to equation (47) and using the relation (48)[11], we obtain the expression of the optimal α_t in terms of J and Φ:

$$
\begin{aligned}
\alpha_t^* &= (\Sigma_t \mathcal{R}_{AA} \Sigma_t^\top)^{-1} \Big(-\frac{J_v v_t}{J_{vv} v_t^2}(\mu_t - R_t \underline{\mathbf{1}}) - \frac{1}{J_{vv} v_t^2} \Sigma_t \mathcal{R}_{AX} G_t^\top J_{vX}\, v_t \\
&\qquad + \frac{J_v v_t + J_{vv} v_t^2}{J_{vv} v_t^2} \sigma_I \Sigma_t \mathcal{R}_{AI} \Big) \qquad (50) \\
&= (\Sigma_t \mathcal{R}_{AA} \Sigma_t^\top)^{-1} \Big(\underbrace{\frac{1}{\gamma}(\mu_t - R_t \underline{\mathbf{1}})}_{\text{I. } \alpha_t^{(M)}} + \underbrace{\Sigma_t \mathcal{R}_{AX} G_t^\top \frac{\Phi_X}{\Phi}}_{\text{II. } \alpha_t^{(I)}} + \underbrace{(1 - \frac{1}{\gamma}) \sigma_I \Sigma_t \mathcal{R}_{AI}}_{\text{III. } \alpha_t^{(P)}} \Big) \\
&= (\Sigma_t^\top)^{-1} \Big(\frac{1}{\gamma} \mathcal{R}_{AA}^{-1} \Sigma_t^{-1} (\mu_t - R_t \underline{\mathbf{1}}) + \mathcal{R}_{AA}^{-1} \mathcal{R}_{AX} G_t^\top \frac{\Phi_X}{\Phi} - \frac{1-\gamma}{\gamma} \mathcal{R}_{AA}^{-1} \mathcal{R}_{AI} \sigma_I \Big) .
\end{aligned}
$$

We can interpret the optimal portfolio allocation as being determined through the trade-off between the asset risks $\Sigma_t \mathcal{R}_{AA} \Sigma_t^\top$ and the "benefits" denoted as I – III in the parenthesis. The first term I refers to the utility increase due to expected excess return. Clearly $(\Sigma_t \mathcal{R}_{AA} \Sigma_t^\top)^{-1}$I corresponds to the *mean-variance efficient* portfolio. Since it considers only the tradeoff between the expected return and the risk, it is also called the myopic portfolio. The second term II appears only in an intertemporal model where the value function Φ depends on the level of the factors X_t. In this case, a sophisticated portfolio decision can increase utility through the correlation between the asset returns and the factor noise. Merton denoted this the *intertemporal hedging term*. For example, suppose a high interest rate level is favored due to more profit, so $J_r > 0$. For the case $\gamma > 1$ we have $\Phi_r < 0$[12], then the intertemporal hedging

[11] from which we have:

$$
\begin{aligned}
\frac{\partial}{\partial t} J &= -\delta J + \gamma \frac{\Phi_t}{\Phi} J , \\
J_v v &= (1-\gamma) J , \\
J_{vv} v^2 &= (1-\gamma)(-\gamma) J , \\
J_X &= \gamma \frac{\Phi_X}{\Phi} J , \\
J_{vX} v &= (1-\gamma)\gamma \frac{\Phi_X}{\Phi} J , \\
J_{X_i X_j} &= \Big(\gamma(\gamma - 1) \frac{\Phi_{X_i}}{\Phi} \frac{\Phi_{X_j}}{\Phi} + \gamma \frac{\Phi_{X_i X_j}}{\Phi} \Big) J .
\end{aligned}
$$

[12] This result is easily shown by taking the derivative of the both sides of equation (49).

term in equation (50) will suggest to increase the holding in an asset whose return shock is negatively correlated with interest rate shocks. For example, the intertemporal hedging term will suggest to investors to increase the bond holding if the return shock of the bond is negatively correlated with the interest rate shock, as is usually the case. The third term III is due to the stochastic price index so we call this the *inflation hedging term*. In Brennan and Xia (2002) and Munk et al. (2004) we can also find the same decomposition of the optimal portfolio.

Applying the expression (50) to the HJB equation (47), the HJB equation is transformed into the form

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Phi + F_t^\top \Phi_X \\ &+ \Big(\frac{1-\gamma}{\gamma} G_t \mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\Sigma_t^{-1}(\mu_t - R_t\underline{\mathbf{1}}) - \frac{(1-\gamma)^2}{\gamma} G_t \mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI}\sigma_I - (1-\gamma) G_t \mathcal{R}_{XI}\sigma_I\Big)^\top \Phi_X \\ &+ \frac{1}{2}\sum_{i,j=1}^n \Phi_{X_i X_j} G_{it}\mathcal{R}_{XX}\Sigma_{jt}^{X\top} \\ &+ \frac{1-\gamma}{2\Phi}\sum_{i,j=1}^n \Phi_{X_i}\Phi_{X_j}\big(G_{it}\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AX}\Sigma_{jt}^X - G_{it}\mathcal{R}_{XX}\Sigma_{jt}^X\big) \\ &+ \Phi\Big(-\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}(R_t - \pi_t + \sigma_I^2) + \frac{1-\gamma}{2\gamma^2}(\mu_t - R_t\underline{\mathbf{1}})^\top(\Sigma_t\mathcal{R}_{AA}\Sigma_t^\top)^{-1}(\mu - R_t\underline{\mathbf{1}}) \\ &\quad + \frac{(1-\gamma)^3}{2\gamma^2}\sigma_I^2\mathcal{R}_{IA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI} - \frac{(1-\gamma)^2}{\gamma^2}(\mu_t - R_t\underline{\mathbf{1}})^\top\Sigma_t^{\top -1}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI}\sigma_I - \frac{1-\gamma}{2}\sigma_I^2\Big) . \end{aligned} \tag{51}$$

3.4 Solving for the Intertemporal Portfolio

In general, if the factor innovation W_t^X is a subset of the asset return risk W_t, then we can obtain

$$\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AX} = \mathcal{R}_{XX} ,$$

and the nonlinear term in the fourth line in (51) becomes zero. As a consequence, the HJB equation (51) reduces to a linear second order PDE and we can use the Feymann-Kac formula as shown in the Appendix to solve the HJB equation. [13] This is exactly the case for our asset

[13] This reduction can be also found in Liu (2005) in the case without inflation risk.

model and the solution for $\Phi(t,T,X_t)$ is then given by

Property 3

$$\Phi(t,T,r_t,\pi_t) = e^{\frac{1-\gamma}{\gamma}B_r(T-t)r_t}\Psi(t,T), \tag{52}$$

where

$$\begin{aligned} & \Psi(t,T) \\ = \;& \exp\Big(j(T-t) + \frac{1-\gamma}{\gamma}\big(T-t-B_r(T-t)\big)\big(\bar{r} + \hat{z}_1\frac{g_r}{\kappa_r}\big) \\ & + \frac{1}{2}\big(\frac{1-\gamma}{\gamma}\big)^2\big(\frac{g_r}{\kappa_r}\big)^2\big(\,T-t-2B_r(T-t) + \frac{1-e^{-2\kappa_r(T-t)}}{2\kappa_r}\,\big)\Big)\,, \end{aligned}$$

where

$$j = -\frac{\delta}{\gamma} + \frac{1-\gamma}{2\gamma^2}\lambda^\top \mathcal{R}_{AA}^{-1}\lambda + \frac{(1-\gamma)\sigma_I^2}{2\gamma^2} - \frac{1-\gamma}{\gamma^2}\lambda_I\sigma_I \tag{53}$$

$$z = \frac{1-\gamma}{\gamma}\begin{pmatrix} \lambda_r - \sigma_I\rho_{Ir} \\ \lambda_\pi - \sigma_I\rho_{I\pi} \end{pmatrix}, \tag{54}$$

and

$$B_r(T-t) = \frac{1-e^{\kappa_r(T-t)}}{\kappa_r}\,. \tag{55}$$

The notation $\hat{z}_1$ denotes the first element in $\hat{z}$ where

$$\hat{z} := \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} := \mathcal{C}^{-1}z$$

with $\mathcal{C}$ lower-triangular Cholesky decomposition of $\mathcal{R}_{XX}$ ($\mathcal{C}\mathcal{C}^\top = \mathcal{R}_{XX}$). For this investment environment described above, $W_t^X = \left(W_t^r, W_t^\pi\right)^\top$, so

$$\mathcal{R}_{XX} = \begin{pmatrix} 1 & \rho_{r\pi} \\ \rho_{r\pi} & 1 \end{pmatrix}.$$

After having obtained the value function Φ, we still need to solve for the factor elasticity Φ_X/Φ.

Property 4 *The factor elasticities are given by*

$$\begin{pmatrix} \frac{\Phi_r}{\Phi} \\ \\ \\ \frac{\Phi_\pi}{\Phi} \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma}{\gamma}\frac{B_r(T-t)}{T-t} \\ \\ \\ 0 \end{pmatrix} . \tag{56}$$

Property 4 is proved simply by differentiating $\Phi(t, T, r_t, \pi_t)$ given in (52).

The parameter κ_r here is the mean-reverting parameter for the real interest rate r_t. It should be noted that the value function $\Phi(t, T, r_t, \pi_t)$ does not depend exactly on the level of the expected inflation rate π_t, for which there is a simple explanation in the following. The agents' objective function (38) depends on the real wealth they expect to achieve and the real wealth evolution (42), according to the (no-arbitrage) condition (31), can be rewritten as

$$\begin{aligned} \frac{dv_t}{v_t} &= \big(r_t + \xi_0 + \sigma_I^2\big)dt + \alpha_t^\top \big(\mu_t - R_t\underline{\mathbf{1}} - \sigma_I \Sigma_t \mathcal{R}_I\big)dt \\ &\quad + \alpha^\top \Sigma_t dW_t - \sigma_I dW_t^I \ , \end{aligned}$$

where only the factor r_t appears. In other words, the effect of the expected inflation is absorbed into the real interest rate so only the real interest rate determines the real wealth evolution. A more detailed and technical explanation can be found in the proof of Property 4 in the Appendix.

Applying the result of Property 4 to the optimal portfolio formula (50), we obtain the optimal strategies of the intertemporal investment plan.

Property 5 *The optimal investment proportions are given by*

$$\alpha_t := \begin{pmatrix} \alpha_{1t} \\ \alpha_{2t} \\ \alpha_{3t} \\ \alpha_{4t} \end{pmatrix} = \underbrace{\frac{1}{\gamma}(\Sigma_t^\top)^{-1}\mathcal{R}_{AA}^{-1}\lambda}_{I.\ \alpha_t^{(M)}} + \underbrace{(1-\frac{1}{\gamma})\,(\Sigma_t^\top)^{-1}\begin{pmatrix} -g_r\frac{B_r(T-t)}{T-t} \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{II.\ \alpha_t^{(I)}} + \underbrace{(1-\frac{1}{\gamma})\,(\Sigma_t^\top)^{-1}\begin{pmatrix} 0 \\ 0 \\ \sigma_I \\ 0 \end{pmatrix}}_{III.\ \alpha_t^{(P)}} , \tag{57}$$

where $B_r(T-t)$ is given by (55).

We remark that the order of the investment proportions $(\alpha_{1t}\,\alpha_{2t},\ \alpha_{3t},\ \alpha_{4t})^\top$ is identical with the order in the equation system (34) so that

α_{1t} represents the investment proportion in the nominal bond maturing at T_1,

α_{2t} represents the investment proportion in the nominal bond maturing at T_2,

α_{3t} represents the investment proportion in the IIB maturing at T_3, and

α_{4t} represents the investment the stock,

respectively.

We lay out in more detail the intertemporal hedging term and the inflation hedging term in the following property

Property 6 *The intertemporal and inflation hedging portfolios can be expressed as*

$$\alpha_t^{(I)} = \begin{pmatrix} \mathcal{D}^{-1}B_{n\pi}(\tau_2)B_r(\tau) \\ -\mathcal{D}^{-1}B_{n\pi}(\tau_1)B_r(\tau) \\ 0 \\ 0 \end{pmatrix}, \qquad \alpha_t^{(P)} = \begin{pmatrix} -\mathcal{D}^{-1}B_{n\pi}(\tau_2)B_{rr}(\tau_3) \\ \mathcal{D}^{-1}B_{n\pi}(\tau_1)B_{rr}(\tau_3) \\ 1 \\ 0 \end{pmatrix}, \tag{58}$$

where $\tau = T - t$, $\tau_i = T_i - t$ for $i = 1, 2, 3$ and

$$\mathcal{D} := \det \begin{pmatrix} B_{nr}(\tau_1) & B_{nr}(\tau_2) \\ B_{n\pi}(\tau_1) & B_{n\pi}(\tau_2) \end{pmatrix}. \tag{59}$$

Because all coefficients $B_{**}(\tau_i)$ are positive as given in Property 1, Property 6 implies that the sign of the hedging positions in the intertemporal hedging portfolio $\alpha^{(I)}$ and the inflation hedging portfolio $\alpha^{(P)}$ depend on the sign of the determinant $\mathcal{D}$. We can characterize the conditions for the sign of the determinant $\mathcal{D}$ in Property 7

Property 7 *For $\tau_1 < \tau_2$, we have*

$$\mathcal{D} \begin{matrix} > \\ 0 \\ < \end{matrix} \iff \begin{matrix} \kappa_r > \kappa_\pi \\ \\ \kappa_r < \kappa_\pi \end{matrix}.$$

We define the *conservative* portfolio as the sum of the intertemporal hedging and price hedging terms,

$$\text{Conservative Portfolio} := \alpha_t^{(I)} + \alpha_t^{(P)} \;, \tag{60}$$

so we obtain also the decomposition for the optimal portfolio α_t:

$$\alpha_t \;=\; \frac{1}{\gamma}\,\text{Myopic Portfolio} + (1-\frac{1}{\gamma})\;\text{Conservative Portfolio}\;. \tag{61}$$

The following property is directly obtained by applying Property 6.

Property 8 *The conservative portfolio is given by*

$$\textit{Conservative Portfolio} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

It can be shown easily by adding up the two hedging portfolios.

This result means, in an investment environment with inflation risk, that the most risk averse investors put all their wealth in the IIB which matures at the end of the investment horizon. This is an extension of the case given in Wachter (2003) where the most conservative investors only buy the nominal bond maturing at the end of the horizon when the investment environment is free from inflation risk. Those two results are based on the same intuition that the most conservative investors require a certain payout at the end of the investment. It is clear that the IIB, instead of the nominal bond, guarantees a certain payout when the investment is exposed to inflation risk.

As a comparison we also provide the optimal intertemporal portfolio without an investment opportunity in the IIBs.

Property 9 *The factor elasticities for the intertemporal investment decision without an investment opportunity in IIBs are identical to those given in (56) with an investment opportunity in IIBs.*

Property 9 asserts that the formulae for the factor elasticities for the value function are the same regardless of the inclusion of the IIBs in the investment opportunity set. We can understand this result using the same intuition as for Property 4 and more detailed and technical details are provided in the proof of this Property.

Having obtained the formula of the factor elasticity, the solution of the optimal investment weights is just followed.

Property 10 *The optimal portfolio weights in the case without the investment opportunity in the IIBs are given by*

$$\alpha_t^* = \underbrace{\frac{1}{\gamma}(\Sigma_t^\top)^{-1}\mathcal{R}_{AA}^{-1}\begin{pmatrix}\lambda_r\\ \lambda_\pi\\ \lambda_S\end{pmatrix}}_{I.\ \alpha_t^{(M)}} \underbrace{+(1-\frac{1}{\gamma})(\Sigma_t^\top)^{-1}\begin{pmatrix}-g_r\frac{B_r(T-t)}{T-t}\\ 0\\ 0\end{pmatrix}}_{II.\ \alpha_t^{(I)}} \underbrace{+(1-\frac{1}{\gamma})(\Sigma_t^\top)^{-1}\sigma_I\mathcal{R}_{AA}^{-1}\begin{pmatrix}\rho_{rI}\\ \rho_{\pi I}\\ \rho_{SI}\end{pmatrix}}_{III.\ \alpha_t^{(P)}}, \tag{62}$$

where $B_r(T-t)$ as same as (55).

Without the investment opportunity in the IIBs, the risk of the stochastic price index W_t^I can only be hedged by its correlations with the other risky assets, as shown in the third term III.$\alpha^{(P)}$ in (62). Without the IIBs, the financial market exposed to inflation risk is incomplete, no asset can give a certain payout. Therefore, there is no longer a riskless strategy for the most risk averse agents and they can only partially hedge the systematic risk by utilization of correlations of asset returns.

Since the factor elasticity without IIB as given in Property 9 is same as that with IIB, and since the intertemporal hedging term II.$\alpha^{(I)}$ in the optimal portfolio (62) is closely related to the factor elasticity, we can expect that the intertemporal hedging term in the case without IIB is very similar to that with IIB.

Property 11 *The intertemporal hedging portfolio in the case without IIB can be expressed by*

$$\alpha_t^{(I)} = \begin{pmatrix} \mathcal{D}^{-1} B_{n\pi}(\tau_2) B_r(\tau) \\ -\mathcal{D}^{-1} B_{n\pi}(\tau_1) B_r(\tau) \\ 0 \end{pmatrix}, \tag{63}$$

where $\tau = T - t$, $\tau_i = T_i - t$ for $i = 1, 2$ and $\mathcal{D}$ is defined as (59).

4 Model Estimation

This section undertakes three tasks. The first one is to estimate the parameters which are required to implement the optimal intertemporal portfolio rules described above. The second task is to use the Kalman filter to estimate the instantaneous real interest rate and the instantaneous expected inflation rate that are not directly observed, but are reflected implicitly in the evolution of the real and nominal term structures. The third task is a validation check of the estimated results where the fitting errors of the market data should be small and the estimation results should be economically reasonable.

The US Treasury provides daily data of real bond yields from 2003. These data allow us to estimate the term structure in a new way. We can estimate the instantaneous real interest rate directly from the market real yield data, whereas the conventional way of estimating the real interest rate would require us to first estimate the expected rate of inflation. Once the real interest rate has been estimated, we can utilize nominal bond yield data, which are considered to bear inflation risk, to estimate the expected rate of inflation. This estimation procedure has the advantage that although our nominal term structure has two unobservable state variables,

r_t and π_t, we can still identify them and estimate them through the market data.

We set one time unit equal to one year. The time interval for daily data is 1/250 and for monthly data 1/12.

4.1 The Term Structure of Real Yields

The real yield data are calculated based on the market returns of the Treasury inflation-protected securities (TIPS) using the cubic spline method.[14] Our data consist of daily real yields with maturity horizons 5, 7, and 10 years from Jan. 02, 2003 until May 31, 2005 containing 603 observations in all. The time series of these yields are displayed in Fig. 1. We employ the Kalman filter to estimate the factor X_t from the US data of the real yields. By implementing the Kalman filter[15], the *observation equation* is the real yield formula (16), where the coefficients $A_r(\tau)$ and $B_{rr}(\tau)$ have been solved and are given by (30) and (29), with measurement errors. Thus, the observation equation here is given by

$$Y_r(t, t+\tau, r_t) = \frac{A_r(\tau)}{\tau} + \frac{B_{rr}(\tau)}{\tau} r_t + \epsilon_t^\tau \, , \tag{64}$$

where ϵ_t^τ denotes the measurement error which is assumed to be independently and identical normal distributed with mean 0 and variance σ_{ϵ^τ}. The state equation here is obtained by discretizing factor dynamics of r_t (3) using the Euler-Maruyama scheme. The discretized process should be very close to the continuous-time process because the discretization interval is 0.004 corresponding to one day.

The results of the parameter estimation are given in Table 1 and the estimated real interest rate r_t is plotted in Fig. 1.

[14]http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/

[15]See Appendix

Log Likelihood = 10056.45

	Estimate	t-Stat.
κ_r	0.1248	7.31
$\overline{r}$	0.0040	0.02
g_r	0.0101	27.51
λ_r^*	-0.5161	-0.22
σ_ϵ^τ	0.0008	49.84

$\lambda_r^* = \lambda_r - \sigma_I \rho_{Ir}$

τ	5Y	7Y	10Y
Mean	1.16%	1.56%	1.90%
SD	0.25%	0.26%	0.23%
$\frac{A_r(\tau)}{\tau}$	1.14%	1.48%	1.89%
$\frac{B_{rr}(\tau)}{\tau}$(Sensitivity)	74%	67%	57%
$\hat{\sigma}_{\epsilon^\tau}$	8.63e-4	5.83e-4	7.76e-4
$\hat{\sigma}_{\epsilon^\tau}/SD$	35.01%	22.52%	33.23%

Table 1: Upper Panel: estimated parameters for the real yield formula and Lower Panel: statistics, fitting errors, and price sensitivities

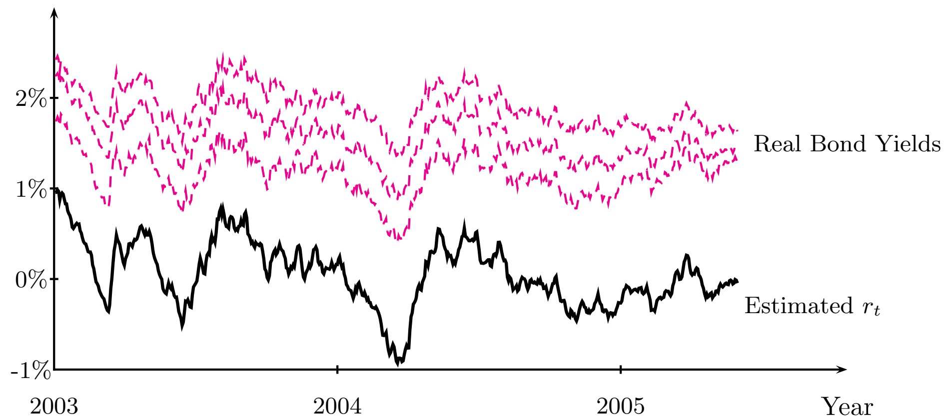

Figure 1: Time Series of Real Yields and the Estimated Real Rate

The average measurement errors of the real yields are given in the last row of Table 1. Compared with the standard deviations of the real yields above, the model can explain around 70% of the variation[16] of the real yields.

The parameter κ_r is related to two features in the real bond model. The first feature is the speed of mean-reversion of the factor r_t as represented in the dynamics (3). The higher this parameter value is the faster the factor r_t comes back to its mean $\overline{r}$ and also the more often the factor crosses the mean. The half decay time of the mean-reverting level κ_r is $(\ln 2)/\kappa_r$. Our estimation result of κ_r in Table 1 gives the half decay time around 5.55 years.

The second feature is the real yield sensitivity with respect to the change of the factor r_t as formulated in the real yield formula (64) where one can see that one unit change of r_t leads to a $\frac{B_{rr}(\tau)}{\tau}(= \frac{1-e^{-\kappa_r\tau}}{\tau})$ unit change of the bond yield $Y_r(t, t+\tau, r_t)$. According the estimation result in the lower penal of Table 1, one unit change of r_t leads to a change of the 5-year real yield by 74% of a unit.

[16]The unexplained fraction is defined as $\frac{\sigma_\epsilon}{\text{SD}}$.

4.2 The Term Structure of Nominal Yields

The market data of nominal yields are also provided by the US Treasury [17] and are calculated based on the market nominal bond returns of Treasury Securities. We take daily nominal yields with time to maturity one month, 3, 6 months, 1 year, 2, 3, 5, 7, 10 and 20 years, also over the horizon Jan. 02, 2003 – May 31, 2005 containing 603 observations. As shown in Fig. 2 the short term nominal yields have an increasing trend after the 2$^{\text{nd}}$ Quarter 2004. During this time, the Federal Open Market Committee (FOCM) conducted a strengthening monetary policy by raising its target interest rate from 1% to 3%. In the same figure we also provide the effective Federal Funds Rate (FFR). The observation equation is based on the yield formula

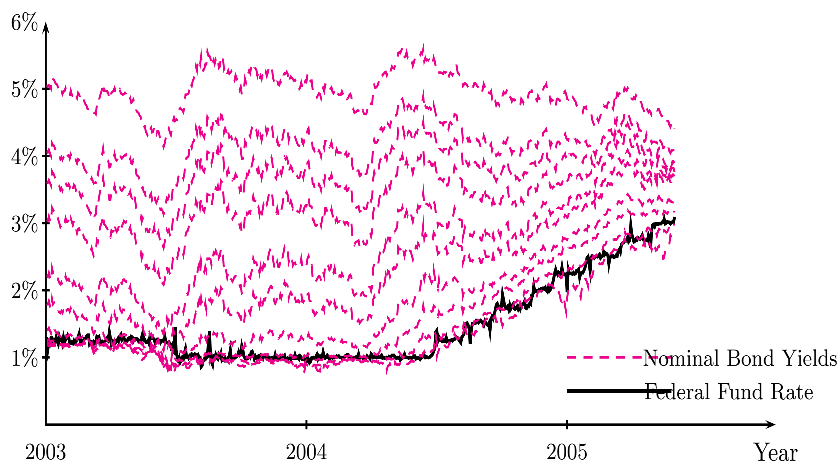

Figure 2: US Nominal Bond Yields and Federal Funds Rate (FFR)

(9) but in addition with the measurement error ϵ_t^τ, thus

$$Y_n(t, t+\tau, r_t, \pi_t) = \frac{A_n(\tau)}{\tau} + \frac{B_{nr}(\tau)}{\tau}\, r_t + \frac{B_{n\pi}(\tau)}{\tau}\, \pi_t + \epsilon_t^\tau \,, \tag{65}$$

where $A_n(\tau)$ is given by (28), $B_{nr}(\tau)$ and $B_{n\pi}(\tau)$ are replaced by (26) and (27), and the measurement errors ϵ_t^τ are identically and independently distributed for all t and τ. For the real interest rate r_t in equation (65) we adopt the previous estimated results because we assume investors in the nominal bond market and the IIB market share the same belief on the instan-

[17] http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/

taneous real rate. The instantaneous inflation expectation π_t, however, is treated as unknown and will be estimated by using the Kalman filter. So, the state equation for implementing the Kalman filter is the discretized dynamics of the expected inflation rate π_t given in (4) by the Euler-Maruyama Scheme. The mean $\overline{\pi}$ is normalized to zero as discussed in Hsiao (2006). The parameters are determined also by using the maximum likelihood method.

The estimation for the correlation coefficient $\rho_{r\pi}$ between the real interest rate shock W_t^r and the expected inflation shock W_t^π requires an iterating estimation scheme due to the following fact. In equation (28) $\rho_{r\pi}$ is a parameter to be determined through the maximum likelihood estimation method. However, after $\rho_{r\pi}$ and all the other parameters have been estimated, we can calculate the *sample correlation coefficient* based on the estimated residuals of (3) and (4), that is

$$\Delta\hat{W}_t^r = \frac{1}{g_r}\big(\Delta r_t - \kappa_r(\overline{r} - r_{t-\Delta})\Delta\big) \,,$$
$$\Delta\hat{W}_t^\pi = \frac{1}{g_\pi}\big(\Delta \pi_t - \kappa_\pi(\overline{\pi} - \pi_{t-\Delta})\Delta\big) \,,$$

and

$$\hat{\rho}_{r\pi} := \mathbf{E}\big[\Delta\hat{W}_t^r \, \Delta\hat{W}_t^\pi\big]/\Delta \,. \tag{66}$$

where $\kappa_r, \overline{r}, \kappa_\pi$ take values of the estimation results. These two estimates for $\rho_{r\pi}$, have to be consistent with each other. However, it is not usually the case. To gap this inconsistency of estimating $\rho_{r\pi}$, we implement the iterating estimation scheme: in the first step we fix $\rho_{r\pi}$ to be a value $\rho_{r\pi}^{(1)}$, say, 0, and estimate all other parameters by the maximum likelihood method and then calculate the estimated sample correlation $\hat{\rho}_{r\pi}^{(1)}$ as given in (66). Next, we compare $\rho_{r\pi}^{(1)}$ and $\hat{\rho}_{r\pi}^{(1)}$, if they are close to each other, we stop the iteration scheme, otherwise we set the initial value $\rho_{r\pi}^{(2)} = \hat{\rho}_{r\pi}^{(1)}$ for the second step and repeat the whole above process. Under the assumption that the estimation model is true and the maximum likelihood estimator is consistent, this iteration scheme provides a consistent estimator.

We implement the above iteration scheme with the initial correlation coefficient $\rho_{r\pi}^{(1)} = 0$. The sample correlation coefficient for the first iteration step is calculated as $\hat{\rho}_{r\pi}^{(1)} = -0.5476$. Taking this value as the correlation coefficient for the second step, the sample correlation coefficient is then calculated as $\hat{\rho}_{r\pi}^{(2)} = -0.5250$. We judge that these two values are closed enough and stop the iteration scheme at the second step.

The estimation results of the parameters are summarized in Table 2. The mean-reverting parameter $\kappa_\pi = 0.4016$ means that the estimated π_t with the dynamics (4) is a stationary process. The estimate corresponds to a half-decay time around 1 and three quarter years (1.73 years). The π_t-sensitivity based on the estimated value is listed with different time to maturity in the lower panel in Table 2. It decreases with the time to maturity. The development of the nominal term structure, which is characterized by the decreasing term premia (the yield spread), can be explained mathematically by the increasing level represented by the term $\frac{A_n(\tau)}{\tau}$ and the decreasing sensitivity to the rising π_t. When the sensitivity goes down, the upward trend contributed by π_t turns flatter as we can see in the time series of the long-term yields in Figure 3.

Figure 3: Nominal Yields and Estimated Factors

In the lower penal of Table 2 we provide the estimate for the scale of the measurement error σ_ϵ for each bond and its relative fitting error $\sigma_\epsilon/\text{SD}$. It is satisfactory for the fitting of the short-term yields, while there is still room for improvement for those of the long-term yields. Figures 4 and 5 plot the estimated and the market nominal yields for one year and ten years maturity respectively.

Log-Likelihood $= 27479.20$

	Estimates	t-stat.
κ_π	0.4016	34.44
g_π	0.0067	65.00
λ_π	-1.5680	-22.36
ξ_0	-0.0012	-4.53
$\hat{\sigma}_\tau$	0.0025	58.87
$\rho_{r\pi}$	-0.5476	

τ	1M	3M	6M	1Y	2Y
Mean	1.37%	1.47%	1.63%	1.84%	2.26%
SD	0.59%	0.66%	0.75%	0.75%	0.75%
$\frac{A(\tau)}{\tau}$	-0.05%	0.08%	0.27%	0.62%	1.23%
$\frac{B_{n\pi}(\tau)}{\tau}$(Sensitivity)	98.34%	95.14%	90.60%	82.36%	68.74%
σ_ϵ	0.31%	0.21%	0.13%	0.13%	0.24%
σ_ϵ/SD	51.79%	31.45%	17.94%	17.08%	31.73%

τ	3Y	5Y	7Y	10Y	20Y
Mean	2.65%	3.31%	3.38%	4.17%	4.95%
SD	0.65%	0.47%	0.37%	0.32%	0.29%
$\frac{A(\tau)}{\tau}$	1.73%	2.52%	3.12%	3.77%	4.95%
$\frac{B_{n\pi}(\tau)}{\tau}$(Sensitivity)	58.12%	43.11%	33.43%	24.45%	12.44%
σ_ϵ	0.26%	0.27%	0.26%	0.25%	0.34%
σ_ϵ/SD	39.80%	57.98%	71.86%	78.07%	117.34%

Table 2: Upper Panel:estimated parameters for nominal term structure; Lower Panel: statistics, fitting errors, and yield sensitivity

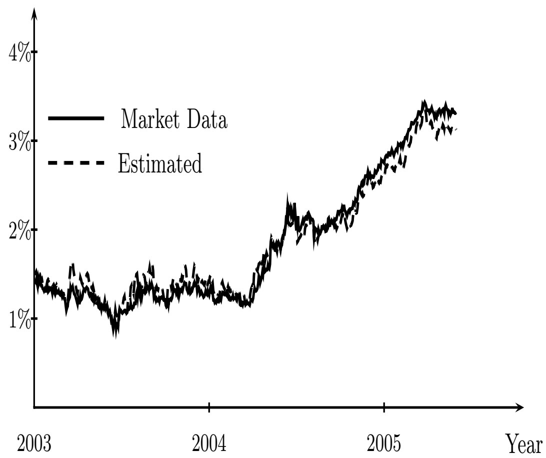

Figure 4: Estimated and Observed 1Y yield

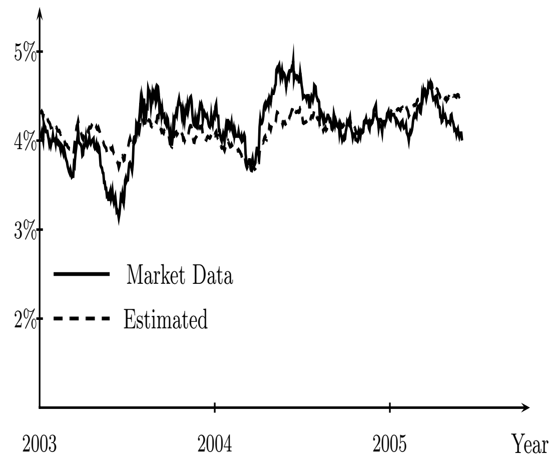

Figure 5: Estimated and Observed 10Y yield

As a validation check for the model estimation, we compare the instantaneous nominal interest rate given by the formula (31) based on the estimation results, and the corresponding market interest rates. We take the Federal Funds rate, which is not included in the model estimation. The comparison is shown in Fig. 6 where we found the fitting is satisfactory after the fourth Quarter 2003.

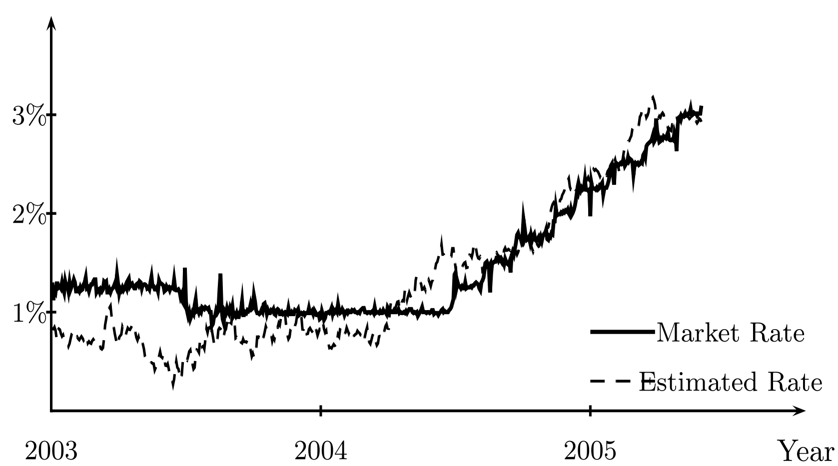

Figure 6: Federal Fund Rate and the Estimated Instantaneous Rate

4.3 Estimation of Realized Inflation Dynamics

We estimate the price index dynamics (1) based on market data. We employ the Consumer Price Index for all urban consumers (CPI-U) provided by he U.S. Department of Labor[18], which are used to adjust the US TIPS.

Using the Itô Lemma, we transform the dynamics (1) into

$$d\ln I_t = (\pi_t - \frac{\sigma_I^2}{2})dt + \sigma_I dW_t^I.$$

Discretising it by using the Euler-Maruyama scheme, we obtain

$$\ln I_{t+\Delta} - \ln I_t = (\pi_t - \frac{\sigma_I^2}{2})\Delta + \sigma_I (W_{t+\Delta}^I - W_t^I)\ , \tag{67}$$

where we assume π_t follows the dynamics (4).

The annualized realized inflation $(\ln I_{t+\Delta} - \ln I_t)/\Delta$ is plotted in Fig. 7. To estimate the

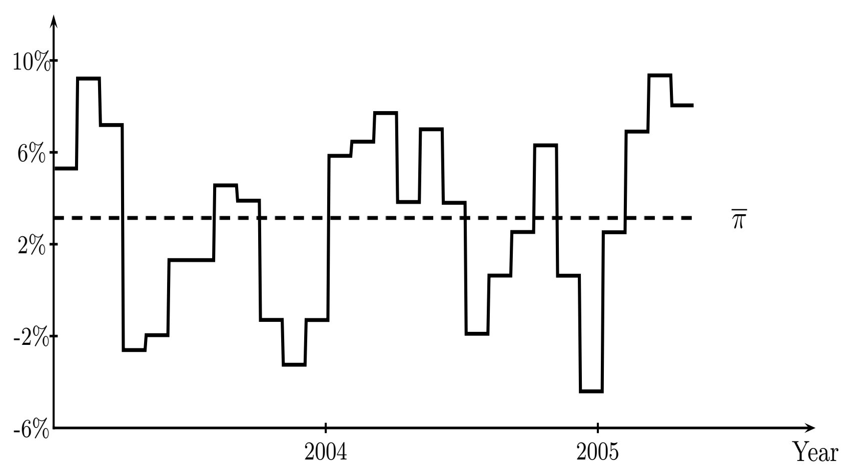

Figure 7: Realized and Filtered Annualized Inflation

unobservable process π_t through the time-discrete observation of the price index I_t, we face a filtering problem as encountered in the previous subsections. We still employ the Kalman filter

[18]http://www.bls.gov/cpi/home.htm

method. In this case, the observation equation is given by the dynamics (67) and the state equation is the dynamics (4) of π_t.

The estimation results are given in Table 3.

	Estimate	t-stat.
κ_π	0.4163	5.38
g_π	0.0000	0.00
$\overline{\pi}$	0.0315	4.18
σ_I	0.0115	11.47

Table 3: Estimation Results for the CPIU

The estimation result $g_\pi = 0.0$ suggests clearly that the underlying factor π_t should remain constant at the level $\overline{\pi} = 3.149\%$ instead of time-varying. We show this the expected $\pi_t = \overline{\pi}$, for all t in Figure 7.

It is worthy remarking that the estimation result for the expected inflation rate π_t here is different from that given in Figure 3 previously based on the nominal term structure model. The variable π_t in the both models incorporates the (instantaneous) inflation expectation. However, along the model context, the estimations for π_t are based on different data set: the estimation here is based on the current realized price index, while the previous estimation in the nominal bond yield formula (9) is based on the nominal and real bond yields with the time maturity stretching from one month until 20 years. Therefore, the variable π_t might have different interpretations. The result given in Figure 7 (constant $\overline{\pi}$) reflects the development of the current price level while the result shown in Figure 3 reflect a long-term development of the market expectation for the inflation. We decide to keep both interpretations for π_t within both contents.

Following the result (32), the market price of the price index risk λ_I is given by

$$\lambda_I = -\frac{\xi_0}{\sigma_I} = \frac{0.0012}{0.0115} = 0.1043\,. \tag{68}$$

Next we calculate the correlation between W_t^I, W_t^r, and W_t^π. We remark that W_t^r and W_t^π are obtained in a daily basis. while the estimated shock W_t^I is in a monthly basis. To calculate $\rho_{Ir}, \rho_{I\pi}$ we accumulate W_t^r and W_t^π to monthly shocks by summing them up.

The sample correlations of the monthly shocks are calculated as $\rho_{Ir} = 0.0609$ and $\rho_{I\pi} = -0.0688$. Both two correlations are quite low.

Having estimated the correlation ρ_{Ir} and using the result for λ_r^* in Table 1, we can calculate the market price of real interest rate risk by $\lambda_r = \lambda_r^* - \sigma_I \rho_{Ir} = -0.5168$.

4.4 Estimation of Stock Return Dynamics

For our intertemporal asset allocation problem, in addition to the bond assets modelled above, we also one stock asset in the investment opportunity set. Applying the Itô formula to the stock price dynamics (33), we obtain one equivalent expression

$$d\ln P_S(t) = \big(R_t + \lambda_S\sigma_S - \frac{\sigma_S^2}{2}\big)dt + \sigma_S dW_t^S\,. \tag{69}$$

The estimation model is obtained by applying the Eular-Maruyama approximation method to the continuous-time dynamics (33) where the discretization interval $\Delta t = 1/250$ for these daily data. The estimation of the parameters in the dynamics (33) is based on data of the daily S&P500 index from Jan. 02 2003 - May 31 2005 including 603 observations, which are plotted in Figure 8. The data can be found in "Finance Yahoo". For the riskless rate R_t we adopt the Federal Funds rate. Figure 9 shows the time series of the daily excess stock returns and Figure

10 shows their distribution.

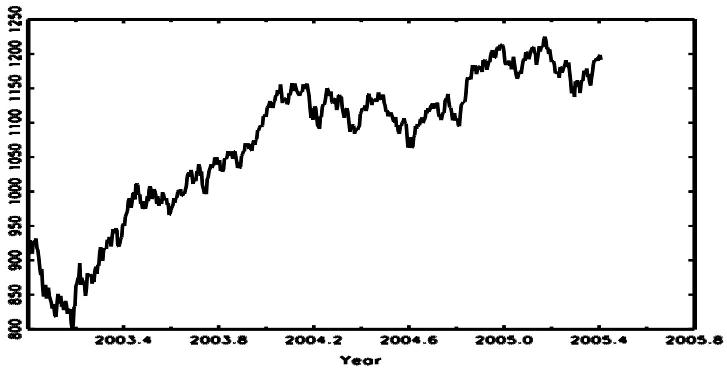

Figure 8: SP500 Index

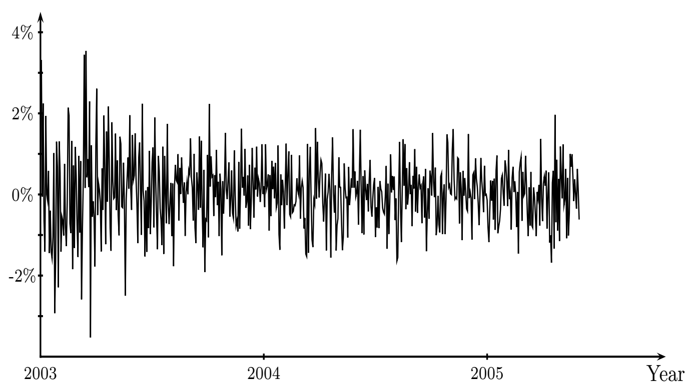

Figure 9: Daily Excess Returns(S&P500)

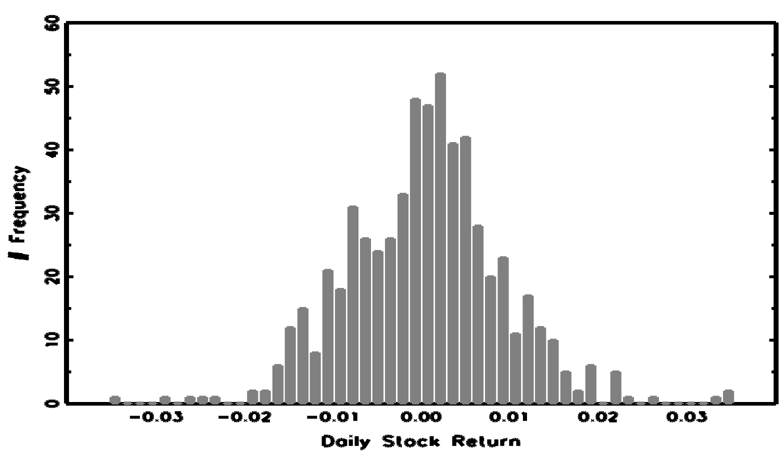

Figure 10: S&P500 Excess Daily Returns Distribution

The parameters in (33) are estimated as $\sigma_S = 0.1391$ and $\lambda_S = 0.8669$.

For the asset allocation problem we still need to know the correlations between the shocks W_t^S and W_t^r, W_t^π and W_t^I. Based on the estimation results, the sample correlations are given by

$$\rho_{Sr} = 0.1744 \quad \rho_{S\pi} = -0.0221 \quad \rho_{SI} = -0.0587 \quad \cdot$$

The correlation between the shocks W_t^S and W_t^I is calculated in a monthly basis.

5 Optimal Portfolios

This section provides concrete investment recommendations for the strategies including investing IIBs. We are interested in studying hedging effect of the IIBs.

We consider for risky assets in the investment opportunity set: a three-year nominal bond (NB3Y), a 10-year nominal bond (NB10Y), a 10-year IIB and a stock whose dynamics of the returns are summarized in (34). The parameter values for this example are adopted from the previous estimation results. We summarize the relevant parameter values for the optimal investment strategies in Table 4.

$$\kappa_r = 0.1241, \quad \overline{r} = 0.0040, \quad g_r = 0.0101$$

$$\kappa_\pi = 0.4016, \quad \xi_0 = -0.0012, \quad g_\pi = 0.0067$$

$$\sigma_S = 0.1391, \quad \sigma_I = 0.0115,$$

$$\lambda_r = -0.5168, \quad \lambda_\pi = -1.5681,$$

$$\lambda_I = 0.1014, \quad \lambda_S = 0.8669,$$

$$\rho_{\pi r} = -0.5082,$$

$$\rho_{Ir} = 0.0609, \quad \rho_{I\pi} = -0.0688,$$

$$\rho_{Sr} = 0.1744, \quad \rho_{S\pi} = -0.0221, \quad \rho_{SI} = -0.0587$$

Table 4: Parameter summary

Figure 11 plots the optimal portfolio weights against the risk aversion parameter $\gamma \in [4, 1000]$. The investment horizon is ten years. In Fig. 11 all positions decrease in absolute value when the agents' risk aversion becomes larger with the only one exception of the IIB. To understand this result we recall the portfolio decomposition (57) and present the weights of each portfolio in Table 5. As the risk aversion γ increases, the optimal portfolio converges to the conservative portfolio as shown in (61). According to Property 8, the conservative portfolio invest all the wealth in the IIB. Further, we look at the intertemporal and inflation hedging portfolios in the conservative portfolio. The sign of the intertemporal hedging position is explained by Properties 6 and 7. In our case we have $\kappa_r < \kappa_\pi$ from the estimation result, so the intertemporal hedging portfolio prefers has long position in the long-term bond and short position in the short-term bond. The exact amounts are given in Table 5.

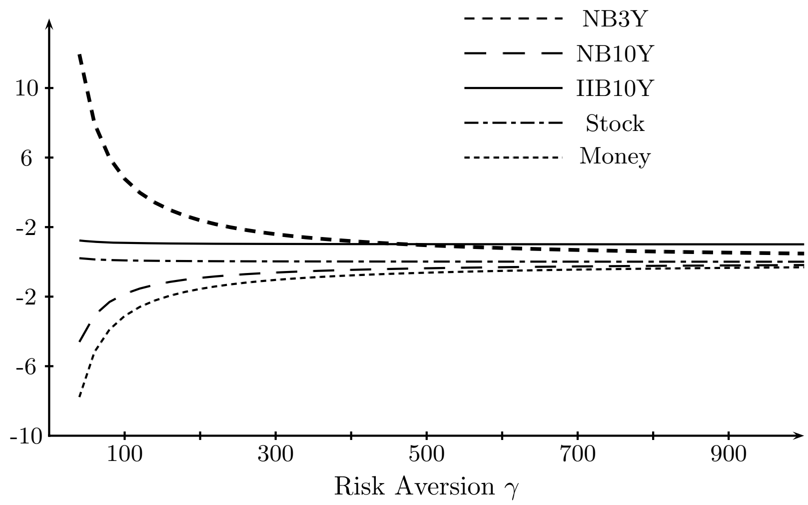

Figure 11: Optimal Portfolio Weights, with IIB

	I.$\alpha^{(M)}$	II.$\alpha^{(I)}$	III.$\alpha^{(P)}$	Conserv
NB3Y	477.72	-3.64	3.64	0.00
NB10Y	-184.27	2.59	-2.59	0.00
IIB10Y	10.20	0.00	1.00	1.00
Stock	8.42	0.00	0.00	0.00
Money	-311.08	2.04	-1.04	0.00

Table 5: Decomposition of Portfolio with IIB

Table 5 shows that the myopic portfolio I. $\alpha^{(M)}$ have very extreme positions for the two nominal bonds. This might be explained by the hight correlations between the bonds are quite high as given in

$$\text{Cor(NB3,NB10)} = 0.92 \quad \text{Cor(NB3,IIB10)} = 0.81 \quad \text{Cor(NB10,IIB10)} = 0.97 \;\cdot$$

The high correlation between the two nominal bond provide an excellent condition to get rid the return risk by a "long one and short the other" strategy. Although the IIB is also highly correlated with the long-term nominal bond, it has a more moderate position as given in Table 5 because IIB is not only considered for hedging the return risk but also for hedging (realized) inflation risk.

The optimal portfolio strategies without the opportunity to invest in IIBs are shown in Fig. 12. The message from the figure is clear: without the investment opportunity in IIBs, more risk averse agents go back to demand the long-term bond.

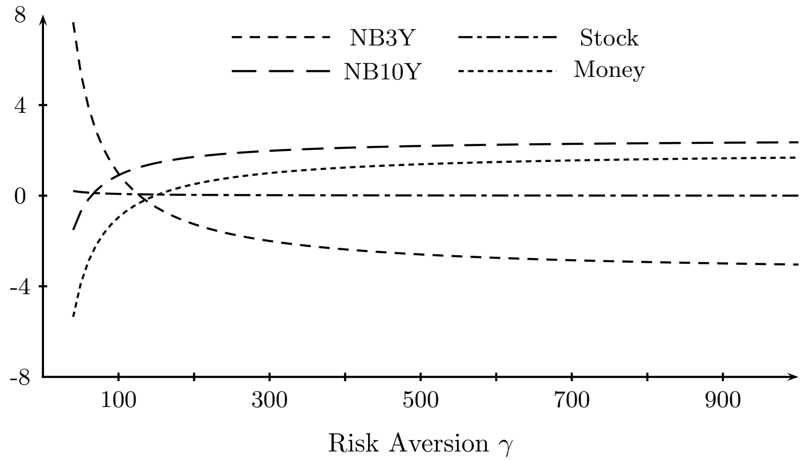

Figure 12: Optimal Portfolio Weights, without IIB

We give exact values of each portfolios in Table 6.

	I/Myopic	II	III	Conserv
NB3Y	442.17	-3.64	0.151	-3.49
NB10Y	-115.60	2.59	-0.076	2.51
Stock	8.36	0.00	-0.006	-0.006
Cash	-290.94	2.04	0.931	1.97

Table 6: Portfolio Decomposition without IIB

Comparing between the intertemporal and the inflation hedging portfolios, the first one dominates in the conservative portfolio. The intertemporal hedging portfolio has a long position in the long-term bond and a short-position in short-term bond because $\kappa_r < \kappa_\pi$ according to Property 10 and Property 7. Recall Property 11, the holding amounts the two nominal bonds in the intertemporal hedging portfolio are just the same as those in the case with IIB given in Property 6. The inflation hedging portfolio is relatively weak where without IIBs agents can only hedge the (realized) inflation risk through the correlation between asset returns and the price index change.

Both two examples in our imtertemporal framework, with and without IIBs, can explain the investment puzzle raised by Canner, Mankiw and Weil (1997) where the bond-to-stock ratio increases with risk aversion. In our examples, the stock has no hedging function at all in the case with IIBs and a very week hedging function in the case without IIBs. Therefore the investment portion in stock decreases by increasing risk aversion and the bond-to-stock ratio goes up.

We also like to examine the investment horizon effect. The risk aversion is fixed at $\gamma = 70$ and the investment horizon goes from 4 to 30 years. We let the IIB and the long-term nominal bond maturing when the investment ends. Figures 13 shows that in the case with IIB, positions in absolute value in the both nominal bonds decrease when the investment horizon increases, while those in the IIB and stock remain constant. This fact can be explained by using the formula for the optimal portfolio given in (57) and letting T approach infinity. We can also obtain the limit positions $\overline{\alpha}_i$ where $\tau_2 = \infty, \tau_3 = \infty$ and they are given by

$$\overline{\alpha}_1 = 5.16 \quad \overline{\alpha}_2 = -1.42 \quad \overline{\alpha}_3 = 1.13 \quad \overline{\alpha}_4 = 0.12 \quad \overline{\alpha}_5 = -3.99 \ \cdot$$

The horizon effect for the case without IIB is shown in Figure 14. The amount of demanding short-term bond decreases when the horizon increases. The stock demand is still kept as constant while the position of the long-term bond turns his sigh when the horizon becomes longer. We also provide the limit positions

$\overline{\alpha}_1 = 2.30 \quad \overline{\alpha}_2 = 0.62 \quad \overline{\alpha}_3 = 0.11(\text{stock}) \quad \overline{\alpha}_4 = -2.03(\text{money}).$

Our result is different to that of Brennan and Xia (2002) because they fixed the bond maturity while varying the horizon length.

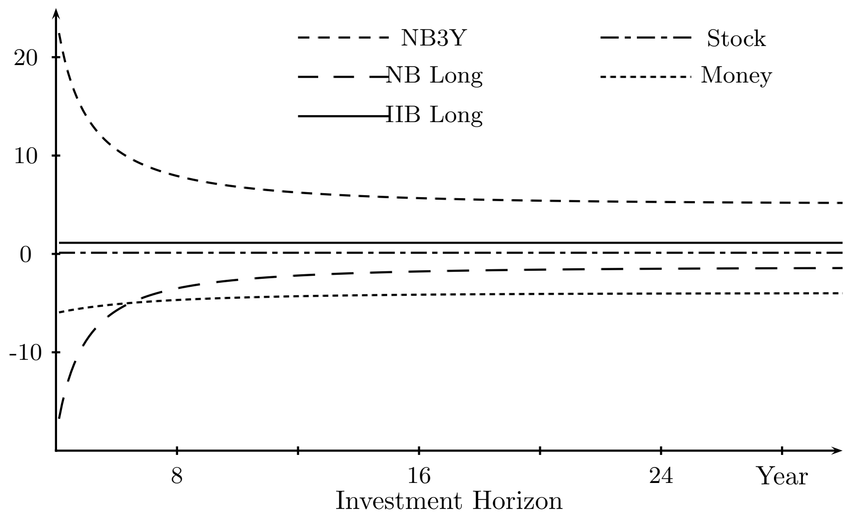

Figure 13: Optimal Portfolio Weights, Horizon Effect, with IIB

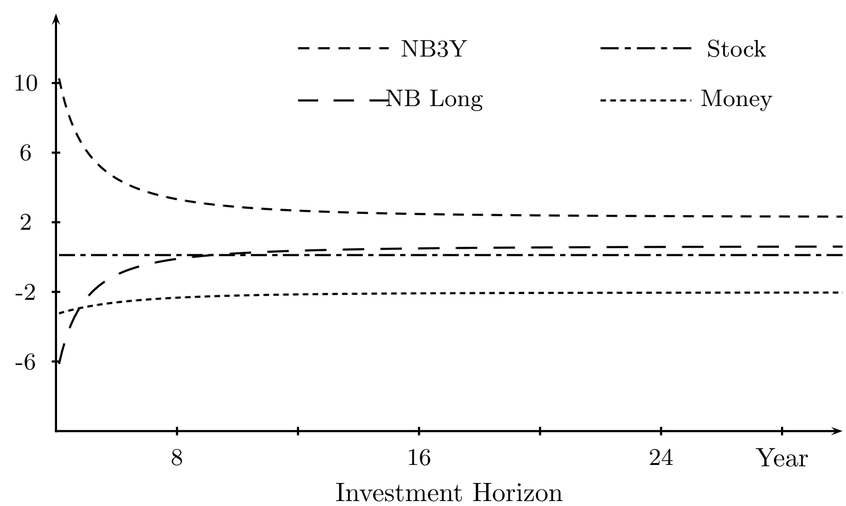

Figure 14: Optimal Portfolio Weights, Horizon Effect, without IIB

6 Conclusion

This paper has considered a multi-factor pricing model for nominal bonds as well as inflation-indexed bonds, and used the classical (nominal) no-arbitrage restriction in order to solve the optimal intertemporal portfolio problem with an investment opportunity including inflation-indexed bonds under inflation risk. We have solved for the optimal intertemporal investment strategies by applying the Feymann-Kac formula and have been able to obtain closed form solutions. In the model calibration analysis, we have presented a new method for estimating the real interest rate without first estimating inflationary expectations. Although there are two unobservable variables in the model, the instantaneous real interest rate and the instantaneous anticipated inflation rate, we have been able to estimate both of them successively with the Kalman filter.

Overall, the risk aversion parameter turns out to be a main characteristic of the intertemporal optimal portfolio. The less risk averse agents are more concerned with the risk-return trade off, while the more risk averse agents prefer certainty of the payout. Hedging strategies are quite different with respect the presence of inflation risk. In a world without inflation risk, the nominal bond maturing at the final day is an ideal hedging asset because it can provides a certain payout when the investment ends, as mentioned in Wachter (2003). However, when the investment is exposed to inflation risk, the role of the long-term nominal bond will be taken over by the IIB maturing at the final day based on the same reasoning. Further, when the IIBs are not available for hedging inflation risk, agents will revert to demanding the long-term bond maturing at the final day in our case.

Similar to the results of Campbell and Viceira (2001), and Brennan and Xia (2002), the positions of the bond holding or the short positions are large, especially in the myopic portfolios.

Such recommendations would not be practical because such an extreme investment strategy sometimes over 100 times of the entire wealth, could not be accepted in real world situations. These observations suggest that future research should focus on the inclusion into the intertemporal optimization problem of real market frictions, such as short-sale constraints, transaction costs, and position limits, in order to obtain investment recommendations within a reasonable range.

7 Appendix

Proof of Property 1

First we prove the second part. Using equation (21) and the no-arbitrage constraints (24) and (25) we have

$$\mu_I(t,\tau) - R_t = \big(\mu_r(t,\tau) + \pi_t - B_{rr}(\tau) g_r \sigma_I \rho_{Ir}\big) - R_t \tag{70}$$

$$\underset{(24)}{=} -\lambda_r B_{rr}(\tau) g_r + \lambda_I \sigma_I \tag{71}$$

$$\Rightarrow \quad -B_{rr}(\tau) g_r \big(\lambda_r - \sigma_I \rho_{Ir}\big) = \mu_r(t,\tau) - \big(R_t - \pi_t + \lambda_I \sigma_I\big)$$

$$\underset{(25)}{=} \mu_r(t,\tau) - r_t \ .$$

Using the definition of μ_r in (19) we rewrite the equation above as

$$\begin{aligned} 0 &= \big(\frac{d}{d\tau} B_{rr}(\tau) + B_{rr}(\tau)\kappa_r - 1\big)\, r_t \\ &\quad + \frac{d}{d\tau} A_r(\tau) - B_{rr}(\tau)(\kappa_r \bar{r} - \lambda_r g_r) + \frac{1}{2} g_r^2 B_{rr}(\tau)^2 \ . \end{aligned} \tag{72}$$

Since r_t is a stochastic process, the equation above holds if and only if

$$\frac{d}{d\tau} B_{rr}(\tau) + B_{rr}(\tau)\kappa_r - 1 = 0 \ , \tag{73}$$

$$\frac{d}{d\tau} A_r(\tau) - B_{rr}(\tau)(\kappa_r \bar{r} - \lambda_r g_r) + \frac{1}{2} g_r^2 B_{rr}(\tau)^2 = 0 \ . \tag{74}$$

Then, $B_{rr}(\tau)$ is solved as (29) and $A_r(\tau)$ is solved as (30). The solution process can be found, for example in Chiarella (2004) .

The first part the model is of a multi-factor Gaussian model. The solution is similar to the second part. The solution process can be found, for example, for example, in Brigo and Mercurio (2001) .

□

Property 12 *Let $(X_s)_{s\in[0,T]}$ be the solution of the the SDE (43). Let $(z_s)_{s\in[0,T]}$ and $(h_s)_{s\in[0,T]}$ be the processes and $(z_s)_{s\in[0,T]}$ satisfies the Novikov condition*

$$\mathbf{E}\big[\exp\big(\int_0^T z_s^\top \mathcal{R}_{XX}^{-1} z_s ds\big)\big] < \infty\,. \tag{75}$$

Then the function $\Phi(t,T,x)$ satisfying the PDE

$$0 = \frac{\partial}{\partial t}\Phi + \big(F_t + G_t z_t\big)^\top \Phi_X + \frac{1}{2}\sum_{i,j=1}^{n} \Phi_{X_i X_j} G_{it} G_{jt}^\top + \Phi h_t + \epsilon_1\,. \tag{76}$$

and the boundary condition

$$\Phi(T,T,X_T) = 1\,. \tag{77}$$

is given by

$$\Phi(t,T,x) = \mathbf{E}_{t,x}\Big[e^{\int_t^T h_s ds}\Lambda_T + \epsilon_1 \int_t^T e^{\int_t^s h_u du}\Lambda_s ds\Big]\,, \tag{78}$$

where

$$\Lambda_s := \exp\Big(\int_0^s z_u^\top \mathcal{R}_{XX}^{-1} dW_u^X - \frac{1}{2}\int_0^s z_u^\top \mathcal{R}_{XX}^{-1} z_u du\Big)\,, \tag{79}$$

for $s \in [0,T]$. The expectation operator $\mathbf{E}_{t,x}$ takes the expectation with respect to the process $(X_s)_{s\in[0,T]}$ with given initial position $X_t = x$.

Proof see Hsiao (2006).

□

Proof of Property 4

The key of the proof is to apply Property 12 above to the HJB equation (51) which the

$\Phi(t, T, r_t, \pi_t)$ satisfies. Comparing the HJB equation (51) with the formula (76), we can apply Property 12 when we identify the notations by

$$
\begin{aligned}
z_t &= \frac{1-\gamma}{\gamma}\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\lambda - \frac{(1-\gamma)^2}{\gamma}\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI}\sigma_I - (1-\gamma)\mathcal{R}_{XI}\sigma_I\ , && (80)\\
h_t &= \frac{1-\gamma}{\gamma}r_t + j_t\ , && (81)\\
j_t &= -\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma}(\xi_0 + \sigma_I^2) + \frac{1-\gamma}{2\gamma^2}\lambda^\top\mathcal{R}_{AA}^{-1}\lambda && (82)\\
&\quad + \frac{(1-\gamma)^3}{2\gamma^2}\sigma_I^2\mathcal{R}_{IA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI} - \frac{(1-\gamma)^2}{\gamma^2}\lambda^\top\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI}\sigma_I - \frac{1-\gamma}{2}\sigma_I^2\ .
\end{aligned}
$$

The last equation (82) is obtained using the no-arbitrage equality (31).

It is easy to observe that j_t (82) and z_t (80) are actually constants because of the constant market price of risk and constant correlation matrices. To stress this, we omit the subindex t.

An remarkable feature of the solution structure is that the second factor π_t does not appear in the equations (81) and (80) anymore due to the replacement based on the arbitrage equality (25). So we can expect that the value function $\Phi(t, T, r_t, \pi_t)$ will be independent of π_t.

We note in (82) that $\mathcal{R}_{IA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI} = 1$ and $\lambda^\top\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI}\sigma_I = \lambda_I\sigma_I$. This is because

$$
\mathcal{R}_{AA}^{-1}\mathcal{R}_{AX} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{R}_{AA}^{-1}\mathcal{R}_{AI} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \qquad (83)
$$

Recall the matrix $\mathcal{R}_{AA}$ is the correlation matrix of uncertainty sources of the asset returns, which are $W_t^r, W_t^\pi, W_t^I, W_t^S$, and $\mathcal{R}_{AX}$ is that of the asset returns and factors W_t^r, W_t^π, so $\mathcal{R}_{AX}$ consists of the first two columns of $\mathcal{R}_{AA}$ and $\mathcal{R}_{AI}$ is exactly the third columns of $\mathcal{R}_{AA}$. That explains the equations (83).

Using the matrix identities above to rewrite (82), we can obtain the result (53).

In the expression for z in (80) we have

$$\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\lambda \quad = \quad \begin{pmatrix} \lambda_r \\ \\ \lambda_\pi \end{pmatrix} \ ,$$

and

$$\mathcal{R}_{XA}\mathcal{R}_{AA}^{-1}\mathcal{R}_{AI} \quad = \quad \mathcal{R}_{XI} = \begin{pmatrix} \rho_{rI} \\ \\ \rho_{\pi I} \end{pmatrix} \ .$$

Using these two equalities above we is obtain (54).

Because z is constant, the Radon-Nikodym derivative (79) can be rewritten as

$$\mathbf{E}_t[\Lambda_T] = \exp\left(z^\top \mathcal{R}_{XX}^{-1}(W_T^X - W_t^X) - \frac{1}{2} z^\top \mathcal{R}_{XX}^{-1} z(T-t)\right) \ . \tag{84}$$

Using the notation $\mathcal{C}\mathcal{C}^\top = \mathcal{R}_{XX}$ to rewrite (84) and letting

$$\hat{z} = \mathcal{C}^{-1}z = \begin{pmatrix} \hat{z}_1 \\ \\ \hat{z}_2 \end{pmatrix} \ , \quad \hat{W}_t^X = \mathcal{C}^{-1}W_t^X = \begin{pmatrix} \hat{W}_{1t}^X \\ \\ \hat{W}_{1t}^X \end{pmatrix} \ ,$$

we have

$$\mathbf{E}_t[\Lambda_T] = \exp\left(\hat{z}^\top(\hat{W}_T^X - \hat{W}_t^X) - \frac{1}{2}\hat{z}^\top \hat{z}(T-t)\right) \ .$$

Note that $\hat{W}_t^X$ is an orthogonal Wiener process because $\mathbf{Var}[\hat{W}_1^X] = \mathcal{C}^{-1}\mathcal{R}_{XX}\mathcal{C}^{-1\top} = I_n$.

The solution for r_t is given by[19]

$$r_s = e^{-\kappa_r(s-t)}r_t + \overline{r}(1 - e^{-\kappa_r(s-t)}) + g_r \int_t^s e^{-\kappa_r(s-u)} dW_u^r \ .$$

Using this solution and Fubini's theorem, we calculate

$$\begin{aligned} \int_t^T r_s ds \quad &= \quad (r_t - \overline{r}) \int_t^T e^{-\kappa(s-t)} ds + \overline{r}(T-t) + g_r \int_t^T \int_u^T e^{-\kappa(s-u)} ds dW_u^r \\ &= \quad B_r(t,T)r_t + \overline{r}(T - t - B_r(t,T)) + g_r \int_t^T B_r(u,T) dW_u^r \ , \end{aligned} \tag{85}$$

[19]See for example Kloeden and Platen (1992) .

where

$$B_r(t,T) = \frac{1}{\kappa_r}(1 - e^{-\kappa_r(T-t)}) \ .$$

Summarizing all the above calculations we can rewrite $\Phi(t,T,r_t)$ as

$$\Phi(t,T,r_t) = \mathbf{E}_{t,x}\big[\exp \mathcal{Y}(t,T)\big] \ ,$$

where

$$\begin{aligned} & \mathcal{Y}(t,T) \\ := \ & \frac{1-\gamma}{\gamma}B_r(T-t)r_t + \frac{1-\gamma}{\gamma}\overline{r}\big(T-t-B_r(T-t)\big) + h(T-t) - \frac{1}{2}\hat{z}^\top\hat{z}(T-t) \\ & + \int_t^T (\frac{1-\gamma}{\gamma}g_r B_r(T-u) + \hat{z}_1)d\hat{W}_{1u}^X + \hat{z}_2(\hat{W}_{2T}^X - \hat{W}_{1t}^X) \ . \end{aligned} \tag{86}$$

Note that $\mathcal{Y}(t,T)$ is normally distributed with the mean and the variance given by

$$\begin{aligned} \mathbf{E}_{t,x}[\mathcal{Y}(t,T)] &= \frac{1-\gamma}{\gamma}B_r(T-t)r_t + \frac{1-\gamma}{\gamma}\overline{r}\big(T-t-B_r(T-t)\big) + h(T-t) \\ & \quad -\frac{1}{2}\hat{z}^\top\hat{z}(T-t) \ , \\ \mathbf{Var}_{t,x}[\mathcal{Y}(t,T)] &= \int_t^T (\frac{1-\gamma}{\gamma}g_r B_r(T-u) + \hat{z}_1)^2 du + \hat{z}_2^2(T-t) \ . \end{aligned}$$

Using the equality

$$\mathbf{E}_{t,x}[\exp\big(\mathcal{Y}(t,T)\big)] = \exp\Big(\mathbf{E}_{t,x}[\mathcal{Y}(t,T)] + \frac{1}{2}\mathbf{Var}_{t,x}[\mathcal{Y}(t,T)]\Big) \ ,$$

we obtain the result (52).

□

Proof of Property 5

The result is obtained directly by inserting the model specifications given by (35), (36), (44) and Property 4 into the optimal portfolio solution (50).

□

Proof of Property 6

This property can be easily proved by providing the inverse of the asset volatility matrix $\Sigma_t^\top$ given in (36)

$$(\Sigma_t^\top)^{-1} = \begin{pmatrix} -\frac{B_{n\pi}(\tau_2)}{g_r \mathcal{D}} & \frac{B_{nr}(\tau_2)}{g_\pi \mathcal{D}} & -\frac{B_{rr}(\tau_3)B_{nr}(\tau_2)}{\sigma_I \mathcal{D}} & 0 \\ -\frac{B_{n\pi}(\tau_1)}{g_r \mathcal{D}} & \frac{B_{nr}(\tau_1)}{g_\pi \mathcal{D}} & -\frac{B_{rr}(\tau_3)B_{nr}(\tau_1)}{\sigma_I \mathcal{D}} & 0 \\ 0 & 0 & \frac{1}{\sigma_I} & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_S} \end{pmatrix}$$

where

$$\mathcal{D} := \det \begin{pmatrix} B_{nr}(\tau_1) & B_{nr}(\tau_2) \\ B_{n\pi}(\tau_1) & B_{n\pi}(\tau_2) \end{pmatrix} .$$

□

Proof of Property 9

The proof goes analogously to the proof of Property 4. The difference to the previous proof is that now different correlation matrices $\mathcal{R}_{AA}$, $\mathcal{R}_{AI}$, and $\mathcal{R}_{AX}$ are inserted in the expressions (81), (82) and (80). The asset return innovations have now three sources W_t^r, W_t^π, and W_t^S. The innovation of the price index W_t^I does not appear in the set of asset return uncertainty due to the exclusion of the IIBs.

The substitution of the different correlation matrices leads a change of the constant j and z given in (82) and (80) but not change the basic form given in (81) in terms of the factor r_t. So, the value function in this case will share the same form given in (52) and therefore has the same expression of the factor elasticity (56).

□

Proof of Property 10

The result (62) is obtained simply by inserting the model specific constants into the general solution (50) and then applying the result of Property (9).

□

Proof of Property 11

This property can be easily proved by providing the inverse of the asset volatility matrix $\Sigma_t^\top$

$$(\Sigma_t^\top)^{-1} = \begin{pmatrix} -\frac{B_{n\pi}(\tau_2)}{g_r\mathcal{D}} & \frac{B_{nr}(\tau_2)}{g_\pi\mathcal{D}} & 0 \\ -\frac{B_{n\pi}(\tau_1)}{g_r\mathcal{D}} & \frac{B_{nr}(\tau_1)}{g_\pi\mathcal{D}} & 0 \\ 0 & 0 & \frac{1}{\sigma_S} \end{pmatrix},$$

where $\mathcal{D}$ is given in (59).

□

The Kalman Filter

We employ the maximum likelihood estimation based on the Kalman filter to estimate the real interest rate.

The Kalman filter is applied to a model of *state space expression*[20] which consists of a *measurement equation*

$$y_t = Z_t X_t + d_t + \varepsilon_t \ , \tag{87}$$

and a *transition equation*

$$X_t = T_t X_{t-1} + c_t + R_t \eta_t \ . \tag{88}$$

The variable of interest y_t is observable and is explained by an observable component d_t and *an unobservable state variable* X_t which follows the dynamics (88). The Kalman filter is an algorithm to formulate the best linear projection of X_t on the observed variables y_t and d_t.

[20]See Harvey(1990) or and Hamiltion(1994).

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