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On the Distributional Characterization of Log-returns of a World Stock Index

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Abstract. In this paper we identify distributions which suitably fit log-returns of the world stock index (WSI) when these are expressed in units of different currencies. By searching for a best fit in the class of symmetric generalized hyperbolic distributions the maximum likelihood estimates appear to cluster in the neighborhood of those of the Student t distribution. This is confirmed on a high significance level under the likelihood ratio test.

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1 Introduction

Starting with papers by Mandelbrot (1963) and Fama (1963) and in a subsequent stream of literature, a vast amount of empirical studies estimating the distributions of log-returns for financial securities has been accumulated. Until about the 1987 stock market crash the standard assumption of portfolio managers, financial decision makers and traders was that log-returns are Gaussian distributed for most securities. However, at least after this event it became clear that this is a rough and for certain risk management areas rather dangerous assumption. In reality, extreme log-returns are more probable than suggested by the Gaussian distribution.

Based on a wide range of statistical techniques the majority of authors in the recent literature overwhelmingly agrees on the conclusion that the assumption on normality for log-returns of stocks and exchange rates has to be strongly rejected. The most obvious empirical characteristic, which contradicts the normality assumption, is the large excess kurtosis that is often observed. There is substantial evidence that supports leptokurtic log-return densities, which exhibit heavier tails and are more peaked than would be expected from a Gaussian distribution. A distribution that generally fits log-returns of stock indices has so far not been widely agreed upon.

In the two papers of Markowitz & Usmen (1996a, 1996b), S&P500 log-returns were analyzed statistically in a Baysian framework. They studied twenty years of daily S&P500 data covering the period from 1963 until 1983. Within the well-known family of Pearson distributions, see, for instance, Stuart & Ord (1994), they identified the Student t distribution with about 4.5 degrees of freedom as the best fit to observed daily log-returns of the S&P500 US stock index. The Pearson family includes as special cases, for instance, the normal, chi-square, gamma, beta, Student t, uniform, Pareto and exponential distributions.

An independent empirical study on the log-return distribution of the S&P500 and other stock indices, including those of Switzerland, Germany, Japan and Australia was undertaken in Hurst & Platen (1997). This paper searched in a large family of normal-variance mixture distributions for the best fit of daily stock index log-returns covering the period from 1982 until 1996. The considered family of distributions included many that were suggested by different researchers in the literature, covering the normal, see Samuelson (1957) and Black & Scholes (1973), the alpha-stable, see Mandelbrot (1963), the normal-lognormal mixture, see Clark (1973), the Student t, see Praetz (1972) and Blattberg & Gonedes (1974), the normal inverse Gaussian, see Barndorff-Nielsen (1995), the hyperbolic, see Eberlein & Keller (1995) and Küchler et al. (1995), the variance gamma, see Madan & Seneta (1990) and the symmetric generalized hyperbolic distribution, see Barndorff-Nielsen (1978). This list indicates that many authors proposed a number of important asset price models that correspond to rather different types of log-return distributions. In Hurst & Platen (1997) a maximum likelihood methodology, as described in Rao (1973), was employed to identify the most likely distributions for stock index log-returns. For all major stock indices that were analyzed for the period from 1982–1996 the Student t distribution was here identified as the best fit to the available data with estimated degrees of freedom between 3.0 and 4.5. These results confirm by a different statistical methodology the previously mentioned findings in Markowitz & Usmen (1996b) on S&P500 log-returns. There exist further studies, including Theodossiou (1998), that observed Student t distributed log-returns for asset prices.

The current paper aims to test the Student t conjecture for log-returns of a globally diversified world stock index over the rather long period from 1970 until 2004. To use a world index instead of local stock market indices is motivated by the benchmark approach developed in Platen (2002, 2004c). Instead of a typical statistical analysis of log-returns of stocks, exchange rates or local indices we choose to analyze the log-returns of a global benchmark representing the *world stock index* (WSI). The log-returns of this index are studied for different currency denominations. As described in Platen & Stahl (2003), the denomination of the WSI in units of a given currency reflects the *general market risk* with respect to this currency. We will see that the WSI shows a particular and rather typical log-return distribution. The following study will confirm with high significance that a *Student t distribution* of about *four degrees of freedom* appears to be typical. This agrees with theoretical predictions of Student t distributed log-returns of the WSI with four degrees of freedom, which we describe in the last section.

The paper is organized as follows. Section 2 introduces a class of leptokurtic log-return distributions and forms the WSI. In Section 3 the implemented maximum likelihood ratio test is described. Section 4 tests the densities of log-returns of the WSI in different currencies. Finally, Section 5 derives the minimal market model, see Platen (2002, 2004c), which potentially explains the Student t property of log-returns.

2 A Class of Log-Return Densities

2.1 The World Stock Index as Benchmark

In the following we apply the benchmark approach proposed in Platen (2002, 2004c). As benchmark we use a *world stock index* (WSI), which we construct as a self-financing portfolio of stock market accumulation indices. The weights for the market capitalization chosen are given in the last column of Table 1 which approximate the market capitalization of the included markets. There are only minor changes in the estimated parameters if one uses different weightings as long as the index is broadly diversified. For instance, the MSCI world accumulation index provides very similar results. As shown in Platen (2004b, 2004c), such a diversified portfolio is robust against variations in weightings as long as the

Country	Currency	$\hat{m}_y$	$\hat{\sigma}_y$	$\hat{\beta}_y$	$\hat{\kappa}_y$	Weights
Argentina	ARS	0.000495	0.009705	4.971980	170.049000	0.0023
Australia	AUD	0.000403	0.008815	-0.372951	14.051762	0.0017
Austria	ATS, EUR	0.000367	0.008716	-0.489599	13.406629	0.0161
Belgium	BEF, EUR	0.000332	0.009387	-0.443605	13.231579	0.0061
Brazil	BRL	0.001392	0.011284	2.330662	79.025855	0.0013
Canada	CAD	0.000404	0.007831	-0.579593	18.075123	0.0251
Denmark	DKK	0.000356	0.009526	-0.394520	14.431344	0.0037
Finland	FIM, EUR	0.000396	0.010612	0.238200	70.189255	0.0032
France	FRF, EUR	0.000376	0.009332	-0.416397	13.655084	0.0302
Germany	DEM, EUR	0.000285	0.009417	-0.483043	13.632198	0.0342
Greece	GRD, EUR	0.000569	0.009454	0.841563	35.815507	0.0012
Hong Kong	HKD	0.000421	0.008000	-0.446156	17.986618	0.0231
Hungary	HUF	0.000476	0.008578	-0.184216	17.086200	0.0023
India	INR	0.000542	0.013561	-0.131696	78.892868	0.0047
Indonesia	IDR	0.000571	0.008906	-0.118789	19.917163	0.0201
Ireland	IRP, EUR	0.000373	0.009033	-0.507152	13.844893	0.0055
Italy	ITL, EUR	0.000373	0.008957	-0.535339	16.379118	0.0132
Japan	JPY	0.000238	0.009033	-0.638245	14.427542	0.1550
Korea	KRW	0.000439	0.009636	-0.139465	57.088739	0.0072
Malaysia	MYR	0.000398	0.008500	-0.616024	18.996454	0.0158
Mexico	MXN	0.001152	0.020686	3.860894	278.762521	0.0016
Netherlands	NLG, EUR	0.000300	0.009358	-0.472830	13.914414	0.0193
Norway	NOK	0.000373	0.009196	-0.340799	14.541824	0.0029
Philippines	PHP	0.000460	0.009009	-0.354508	19.339190	0.0041
Portugal	PTE, EUR	0.000391	0.008810	-0.452567	13.152174	0.0013
Singapore	SGD	0.000353	0.007991	-0.554040	17.215279	0.0079
Spain	ESP, EUR	0.000464	0.010334	1.079021	48.935156	0.0124
Sweden	SEK	0.000421	0.009195	0.214018	19.355551	0.0124
Switzerland	CHF	0.000239	0.010097	-0.407918	11.406303	0.0206
Taiwan	TWD	0.000357	0.007900	-0.558432	17.784127	0.0141
Thailand	THB	0.000450	0.009285	0.628255	36.231337	0.0049
Turkey	TRL	0.001029	0.011354	5.010563	155.199258	0.0018
UK	GBP	0.000414	0.009141	-0.482534	13.541676	0.0846
US	USD	0.000374	0.007724	-0.613548	18.808550	0.4301

Table 1: Empirical moments for log-returns of WSI.

weight of each contributing stock remains reasonably small.

We cover with this benchmark more than 95% of the world stock market capitalization for the period from 1970 until 2004. In our study weekends and other nontrading days at the US and European exchanges are excluded.

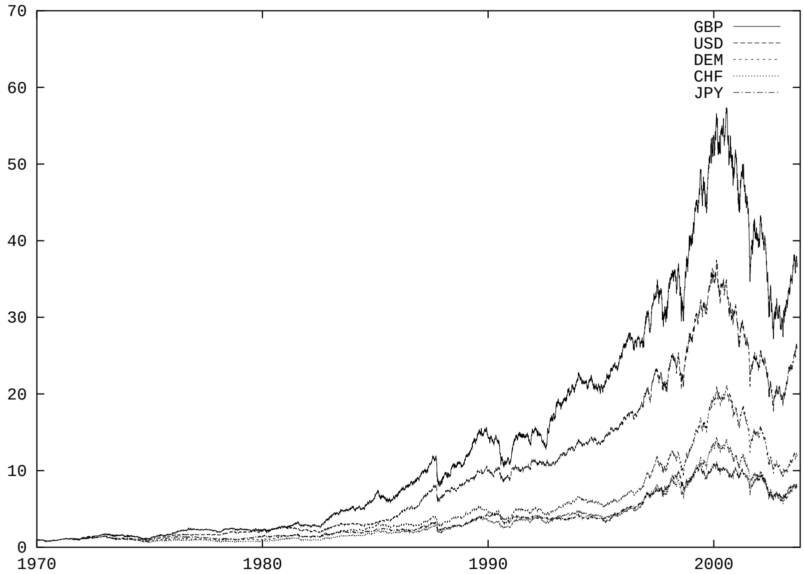

Figure 1: WSI in units of different currencies.

In Figure 1 we plot the resulting WSI for the observation period when denominated in units of US dollar, British Pound, Swiss Franc, Japanese Yen and Deutsche Mark. For convenience, we normalized the initial values to one. We will study the distribution of log-returns of the WSI when denominated in 34 currencies. This will provide some distributional characterization of the general market risk for the respective currency denominations. This is of theoretical significance but also important, for instance, for Value at Risk calculations, see Platen & Stahl (2003).

We deliberately do not adjust for any changes in the parameter over time, market crashes or other influences that may have affected the data. Some methods of data analysis discard extreme values of observations as outliers. But this would be inappropriate in a financial context because it is most important for proper risk management to accurately measure the probability of extreme log-returns. For *daily log-returns* of the WSI for the period from 1970 until 2004 in 34 currency denominations the first four empirical moments yield the average empirical mean $\hat{m}_y = 0.000486$, average standard deviation $\hat{\sigma}_y = 0.009789$, average skewness $\hat{\beta}_y = 0.460316$ and average kurtosis $\hat{\kappa}_y = 44.485182$.

For each currency let us now centralize the log-returns, that is shift these to a zero mean. Furthermore, we scale the resulting log-returns with respect to the variance. The centralized and scaled log-returns of the WSI in different denominations have then zero sample mean and unit sample variance. We then combine all observed centralized and scaled log-returns in one sample. In Figure 2 we show the logarithm of the resulting histogram for the log-returns together with

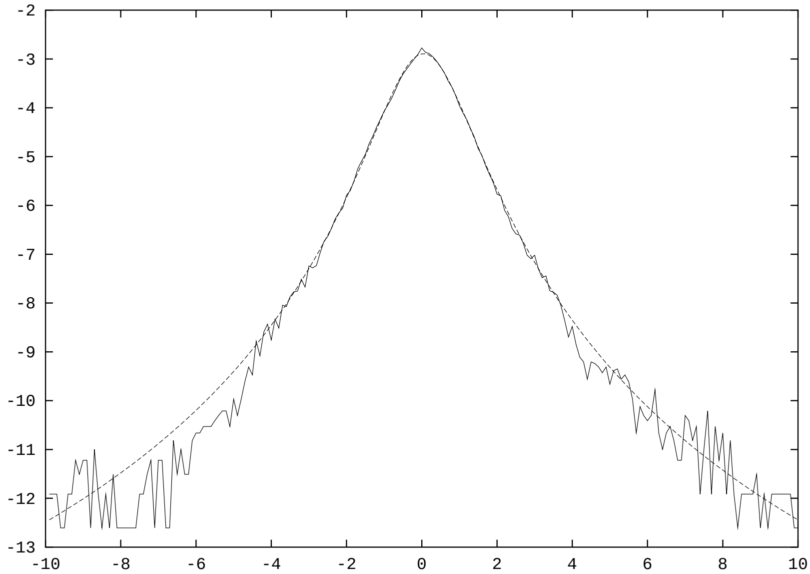

Figure 2: Log-histogram of WSI log-returns.

the logarithm of the Student t density with 3.64 degrees of freedom. One notes the excellent visual fit of this Student t density. Given this good fit it will be our aim to analyze parametrically for each currency denomination the distribution of the log-return in a wide class that contains the Student t distribution.

By considering the relatively small skewness observed in Table 1 the empirical log-return density appears on average still to be fairly symmetrical. The findings in Markowitz & Usmen (1996a, 1996b) for S&P500 log-returns, as well as those in Hurst & Platen (1997) for other stock index log-returns, reported also only a minor skew. Therefore, to simplify our analysis and to focus fully on the identification of the tail properties of the log-return densities we assume that the theoretical densities that we will compare with the empirical density are symmetric, that is we assume, zero skewness. Note that due to the small observation time step size of one day this assumption relates to a higher order effect. It does only marginally influence the empirical results that we obtain. We underline that the following study focuses on the shape of the log-return densities. We avoid to rely on particular moment properties because certain moments may not exist. In particular, the kurtosis of Student t distributed log-returns with four degrees of freedom is known to be infinite.

2.2 Normal-Variance Mixture Densities

For capturing the typical features of log-return distributions let us study a class of normal-variance mixture densities, see Feller (1968), which allows us to keep the empirical analysis rather general. We model the daily *log-return* y_{t_i} of a WSI denominated in a given currency at the ith observation time t_i as

$$y_{t_i} = \mu_{t_i}\,\Delta + \sigma_{t_i}\,\sqrt{\Delta}\,\xi_i, \tag{2.1}$$

$i \in \{0, 1, \ldots\}$. For the small length Δ of typically one day of the observation time interval the first term on the right hand side of equation (2.1) has on average a much smaller absolute value than those that appear typically in the second term. Since the first term is of higher order with respect to Δ, and therefore not of much relevance to our study, we assume, for simplicity, a constant mean parameter $\mu_{t_i}\Delta = \mu$ on the log-returns. It does not change much our empirical results if we set μ simply equal to zero.

To model a normal-variance mixture density for the log-return y_{t_i} we use for all $i \in \{0, 1, \ldots\}$ an independent identically distributed random volatility $\sigma_i = \sigma_{t_i}\sqrt{\Delta}$, together with some independent, standard Gaussian distributed random variable ξ_i. The squared volatility σ_i^2, $i \in \{0, 1, \ldots\}$, is assumed to be distributed according to a given density f_{σ^2}. The generality of the resulting class of normal-variance mixture log-return densities follows from the freedom to adjust the squared volatility density f_{σ^2}. One obtains the normal-variance mixture density function of the log-return y_{t_i}, $i \in \{0, 1, \ldots\}$, in the form

$$f_y(x) = \frac{1}{\sqrt{2\,\pi}} \int_0^\infty \frac{1}{\sqrt{u}} \exp\left\{ -\frac{(x-\mu\,)^2}{2\,u} \right\} f_{\sigma^2}(u)\,du \tag{2.2}$$

for $x \in \Re$ as long as this integral exists. The log-return has then the mean $m_y = \mu$, the variance $v_y = m_{\sigma^2}$, the skewness β_y and the kurtosis

$$\kappa_y = 3\left(1 + \frac{v_{\sigma^2}}{m_{\sigma^2}^2}\right). \tag{2.3}$$

Here the squared volatility σ_i^2, $i \in \{0, 1, \ldots\}$, has mean m_{σ^2} and variance v_{σ^2}.

Samuelson (1957), Osborne (1959) and subsequently many other authors have modeled asset price increments by lognormal random variables, where the log-returns are modeled by Gaussian random variables. The corresponding Gaussian log-return density for the lognormal model results from (2.2) when the density of the squared volatility degenerates to that of a constant $\sigma_i^2 = c^2$, $i \in \{0, 1, \ldots\}$. The Gaussian density is a two parameter density, where μ is a location parameter and c is a scale parameter. The Gaussian log-return hypothesis has been clearly rejected in the literature for indices, exchange rates and equities by a variety of statistical methods. Therefore, we avoid any further consideration of a Gaussian log-return density and analyze instead a rich class of analytically tractable densities covering a wide range of possible tail shapes.

2.3 Symmetric Generalized Hyperbolic Density

Noticeably, many authors have proposed important log-return models involving the class of generalized hyperbolic densities. This class of densities was extensively examined by Barndorff-Nielsen (1977, 1978) and Barndorff-Nielsen & Blaesild (1981). Since we have assumed zero skewness we consider in the following the *symmetric generalized hyperbolic* (SGH) density as a possible log-return density. This density results when the density of the squared random volatility σ_i^2, $i \in \{0, 1, \ldots\}$, is a generalized inverse Gaussian density.

One can show by (2.2) and (2.5) that the SGH density function of the log-return y_{t_i} is

$$f_y(x) = \frac{1}{\delta\, K_\lambda(\alpha\,\delta)} \sqrt{\frac{\alpha\,\delta}{2\,\pi}} \left(1 + \frac{(x-\mu)^2}{\delta^2}\right)^{\frac{1}{2}\left(\lambda - \frac{1}{2}\right)} K_{\lambda-\frac{1}{2}}\!\left(\alpha\,\delta \sqrt{1 + \frac{(x-\mu)^2}{\delta^2}}\right) \tag{2.4}$$

for $x \in \Re$, where $\lambda \in \Re$ and $\alpha, \delta \geq 0$. We set $\alpha \neq 0$ if $\lambda \geq 0$ and $\delta \neq 0$ if $\lambda \leq 0$. The probability density function of σ_i^2 in the normal-variance mixture density (2.2) is here of the form

$$f_{\sigma^2}(x) = \frac{\alpha^\lambda}{2\,\delta^\lambda K_\lambda(\alpha\,\delta)}\, x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\frac{\delta^2}{x} + \alpha^2 x\right)\right\}, \tag{2.5}$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ.

The SGH density is a four parameter density. The two *shape parameters* are λ and $\bar{\alpha} = \alpha\,\delta$, defined so that they are invariant under scale transformations. The parameter μ is a *location parameter*. The other parameters contribute to the scaling of the density. We define the parameter c as the *scale parameter* such that $m_{\sigma^2} = c^2$, that is

$$c^2 = \begin{cases} \frac{2\,\lambda}{\alpha^2} & \text{if } \delta = 0 \text{ for } \lambda > 0,\ \bar{\alpha} = 0, \\ \frac{\delta^2\, K_{\lambda+1}(\bar{\alpha})}{\bar{\alpha}\, K_\lambda(\bar{\alpha})} & \text{otherwise.} \end{cases} \tag{2.6}$$

The variance of σ_i^2 is

$$v_{\sigma^2} = c^4 \left(\frac{K_\lambda(\bar{\alpha})\, K_{\lambda+2}(\bar{\alpha})}{K_{\lambda+1}(\bar{\alpha})^2} - 1\right). \tag{2.7}$$

Consequently, the log-return y_{t_i} has mean $m_y = \mu$, variance $v_y = c^2$, skewness $\beta_y = 0$ and kurtosis

$$\kappa_y = \frac{3\, K_\lambda(\bar{\alpha})\, K_{\lambda+2}(\bar{\alpha})^2}{K_{\lambda+1}(\bar{\alpha})}. \tag{2.8}$$

Furthermore, it can be shown that as $\lambda \to \pm\infty$ or $\bar{\alpha} \to \infty$ the SGH density asymptotically approaches the Gaussian density.

To discriminate between certain candidate log-return densities within the class of SGH densities we will describe in the following four special cases of the SGH density that coincide with the log-return densities of important asset price models.

2.4 Student t Density

Praetz (1972) and Blattberg & Gonedes (1974) proposed for log-returns a Student t density with degrees of freedom $\nu > 0$. This is also the log-return density of the minimal market model derived in Platen (2001, 2002, 2004c), which we derive in Section 5. This density follows from the above SGH density for the shape parameters $\lambda = -\frac{1}{2}\nu < 0$ and $\bar{\alpha} = 0$, that is $\alpha = 0$ and $\delta = \varepsilon\sqrt{\nu}$. Using these parameter values the Student t density function for the log-return y_{t_i} has then the form

$$f_y(x) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\varepsilon\sqrt{\pi\nu}\;\Gamma(\frac{1}{2}\nu)} \left(1 + \frac{(x-\mu)^2}{\varepsilon^2\,\nu}\right)^{-\frac{1}{2}\nu - \frac{1}{2}} \tag{2.9}$$

for $x \in \Re$, where $\Gamma(\cdot)$ is the gamma function. Equation (2.9) expresses the well-known probability density of a Student t distributed random variable with ν degrees of freedom. The squared volatility σ_i^2 is here inverted gamma distributed. The log-return y_{t_i} has mean $m_y = \mu$. The variance $v_y = \frac{\varepsilon^2\,\nu}{\nu-2} = c^2$ is finite only for $\nu > 2$ and the kurtosis $\kappa_y = \frac{3\,(\nu-2)}{\nu-4}$ only exists for $\nu > 4$. The Student t density is a three parameter density. The degree of freedom $\nu = -2\lambda$ is the shape parameter, with smaller ν implying larger tail heaviness for the density. Furthermore, when the degrees of freedom increase, that is $\nu \to \infty$, then the Student t density asymptotically approaches the normal density.

2.5 Normal-Inverse Gaussian Density

Barndorff-Nielsen (1995) proposed log-returns to follow a normal-inverse Gaussian mixture distribution. The corresponding density arises from the SGH density when the shape parameter $\lambda = -\frac{1}{2}$ is chosen. For this parameter value the variance σ_i^2 is inverse Gaussian distributed and it follows by (2.4) that the probability density function of the log-return y_{t_i} is then

$$f_y(x) = \frac{\sqrt{\bar{\alpha}}\,\exp\{\bar{\alpha}\}}{c\,\pi} \left(1 + \frac{(x-\mu)^2}{\bar{\alpha}\,c^2}\right)^{-\frac{1}{2}} K_1\!\left(\bar{\alpha}\sqrt{1 + \frac{(x-\mu)^2}{\bar{\alpha}\,c^2}}\right) \tag{2.10}$$

for $x \in \Re$, where $c^2 = \frac{\delta^2}{\bar{\alpha}}$. Here y_{t_i} has kurtosis $\kappa_y = 3\,(1 + \frac{1}{\bar{\alpha}})$. The normal-inverse Gaussian density is a three parameter density. The parameter $\bar{\alpha}$ is the shape parameter with smaller $\bar{\alpha}$ implying larger tail heaviness. Furthermore, when $\bar{\alpha} \to \infty$ the normal-inverse Gaussian density asymptotically approaches the normal density.

2.6 Hyperbolic Density

Eberlein & Keller (1995) and Küchler et al. (1995) proposed models, where log-returns appear to be hyperbolicly distributed. This occurs for the choice of the shape parameter $\lambda = 1$ in the SGH density. Using this parameter value the probability density function of the log-return y_{t_i} is

$$f_y(x) = \frac{1}{2\,\delta\;K_1(\bar{\alpha})}\,\exp\left\{-\bar{\alpha}\,\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right\} \tag{2.11}$$

for $x \in \Re$, where $\delta^2 = \frac{c^2\,\bar{\alpha}\,K_1(\bar{\alpha})}{K_2(\bar{\alpha})}$. Here y_{t_i} has kurtosis $\kappa_y = \frac{3\,K_1(\bar{\alpha})\,K_3(\bar{\alpha})}{K_2(\bar{\alpha})^2}$. The hyperbolic density is a three parameter density. The parameter $\bar{\alpha}$ is the shape parameter with smaller $\bar{\alpha}$ implying larger tail heaviness. Furthermore, when $\bar{\alpha} \to \infty$ the hyperbolic density asymptotically approaches the normal density.

2.7 Variance Gamma Density

Madan & Seneta (1990) and Geman, Madan & Yor (2001) proposed log-returns to be distributed with a normal-variance gamma mixture distribution. This case is obtained when the shape parameters are such that $\lambda > 0$ and $\bar{\alpha} = 0$, that is, $\delta = 0$ and $\alpha = \frac{\sqrt{2\lambda}}{c}$. With these parameter values the variance σ_i^2 is gamma distributed and the probability density function of y_{t_i} is

$$f_y(x) = \frac{\sqrt{\lambda}}{c\sqrt{\pi}\,\Gamma(\lambda)\,2^{\lambda-1}}\,\left(\sqrt{2\,\lambda}\,\frac{|x-\mu\,|}{c}\right)^{\lambda-\frac{1}{2}}\,K_{\lambda-\frac{1}{2}}\left(\sqrt{2\,\lambda}\,\frac{|x-\mu\,|}{c}\right) \tag{2.12}$$

for $x \in \Re$. Here y_{t_i} has kurtosis $\kappa_y = 3\,(1+\frac{1}{\lambda})$. The variance gamma density is a three parameter density. The parameter λ is the shape parameter with smaller λ implying larger tail heaviness. Furthermore, when $\lambda \to \infty$ the variance gamma density asymptotically approaches the normal density.

3 Maximum Likelihood Ratio Test

The class of SGH densities that we introduced in Section 2.3 represents a rich class of leptokurtic densities. To reject on a given significance level the hypothesis that any of the four previously described SGH densities is not the true underlying density we need to perform a proper statistical test. To discriminate between the various candidate densities we proceed in a well-established manner by using the classical maximum likelihood ratio test, see Rao (1973). We define the *likelihood ratio* in the form

$$\Lambda = \frac{\mathcal{L}_{\mathrm{model}}}{\mathcal{L}_{\mathrm{SGH}}}. \tag{3.1}$$

Here $\mathcal{L}_{\mathrm{model}}$ represents the maximized likelihood function of a given specific nested log-return density, for instance, the Student t density. With respect to this density the maximum likelihood estimate for the parameters have been computed and are then used to obtain the corresponding *likelihood function* $\mathcal{L}_{\mathrm{model}}$. On the other hand, $\mathcal{L}_{\mathrm{SGH}}$ denotes the maximized likelihood function for the SGH density, which is the nesting density that has been similarly obtained. It can be shown, see Rao (1973), that the *test statistic*

$$L_n = -2\,\ln(\Lambda) \tag{3.2}$$

is for increasing number of observations $n \to \infty$ asymptotically chisquare distributed. Here the degrees of freedom equal the difference between the number of parameters of the nesting density and the nested density. The nesting density, which is the SGH density, is a four-parameter density and the nested densities are the Student t, normal-inverse Gaussian, hyperbolic and variance-gamma densities, which are three-parameter densities. Therefore, in the above cases L_n is for $n \to \infty$ asymptotically chisquare distributed with one degree of freedom.

It can be asymptotically shown as $n \to \infty$ that

$$P\left(L_n < \chi^2_{1-\alpha,1}\right) \approx F_{\chi^2(1)}\left(\chi^2_{1-\alpha,1}\right) = 1 - \alpha, \tag{3.3}$$

where $F_{\chi^2(1)}$ denotes the chisquare distribution with one degree of freedom and $\chi^2_{1-\alpha,1}$ is its $100(1-\alpha)\%$ quantile. One can then check, say, for a 99% *significance level* whether or not the test statistic L_n is in the 1% quantile of the chisquare distribution with one degree of freedom. If the relation

$$L_n < \chi^2_{0.01,1} \approx 0.000157 \tag{3.4}$$

is not satisfied, then we reject on a 99% significance level the hypothesis that the suggested density is the true underlying density. To be more accurate one can, if possible, also compare the test statistic L_n with the 0.1% quantile of the chisquare distribution with one degree of freedom, where

$$L_n < \xi^2_{0.001,1} \approx 0.000002. \tag{3.5}$$

When (3.5) is satisfied, then on a 99.9% significance level one cannot reject the considered hypothesis.

The above maximum likelihood methodology offers a natural definition of a *best fit*. We call the density with the smallest test statistic L_n the best fit in the given class of SGH densities. This density maximizes the likelihood ratio Λ given in (3.1).

Country	Student t	Inverse Gaussian	Hyperbolic	Variance Gamma	Degrees of Freedom
Argentina	0.000000	137.566726	377.265316	414.030946	3.215953
Australia	0.000000	24.403096	54.527486	73.900060	4.618187
Austria	1.272478	10.533450	43.923596	63.696828	4.177998
Belgian	0.000000	15.755730	41.664530	59.922270	4.639238
Brazil	0.000000	132.348986	430.843576	444.117392	2.937178
Canada	0.000000	42.829996	80.784298	110.357598	4.927523
Denmark	0.000000	29.607334	72.807340	96.852868	4.328256
Finland	0.000000	130.807532	286.692740	326.546792	3.707350
France	0.303708	12.578282	42.737860	61.753332	4.329017
Germany	0.000000	17.945998	45.546996	64.739532	4.620013
Greece	0.000000	60.072066	120.786456	150.578024	4.340990
Hong Kong	0.000000	31.399542	84.191740	111.611116	4.080899
Hungary	0.002812	33.488642	102.407366	132.556744	3.827598
India	0.000000	218.752194	1096.862148	962.798700	2.283747
Indonesia	0.000000	54.595328	121.131360	148.098694	4.062652
Ireland	0.031796	16.660834	53.818850	76.062630	4.187040
Italy	0.002606	19.207820	60.267448	83.332082	4.172936
Japan	0.000000	24.017652	60.214094	81.351358	4.392711
Korea S.	0.000000	129.955438	386.626152	425.311040	3.265655
Malaysia	0.000000	56.525498	149.299592	189.659002	3.786499
Mexico	0.000000	440.818300	2132.850298	1746.118774	2.207160
Netherlands	0.000000	15.802518	41.848016	60.873070	4.611005
Norway	0.000000	27.920608	71.785758	96.835862	4.256095
Philippine	0.017290	52.048754	167.407546	199.781080	3.458544
Portugal	1.582056	13.129484	54.154638	76.914946	4.071114
Singapore	0.000000	30.656496	73.326620	99.354034	4.396040
Spain	0.000000	70.602362	139.884600	165.163122	4.206288
Sweden	0.000000	66.852560	130.642934	166.827332	4.468983
Switzerland	0.144462	15.390592	46.179620	67.726852	4.471172
Taiwan	0.000000	33.290522	82.900518	110.434774	4.224246
Thailand	0.000000	100.851126	282.814096	314.709524	3.298124
Turkey	0.000000	152.625500	493.285862	506.466162	2.893205
UK	0.000000	21.124390	47.980512	68.613654	4.868241
US	0.000000	31.352956	75.539480	102.259640	4.323809

Table 2: The L_n test statistic for log-returns of the WSI in different currencies.

4 World Stock Index in Different Currencies

We study the log-returns of the above constructed world stock index when denominated in units of major currencies. We use daily data from 1970 until 2004 provided by Thomson Financial. In Table 2 we display the L_n test statistics for log-returns of the WSI in 34 different currency denominations. It is rather apparent that the Student t density shows in all cases the smallest test statistic. For 25 of the 34 currencies one cannot reject on a 99.9% significance level the hypothesis that the Student t density is the true density. The inverse Gaussian density seems to be the second best choice but can be rejected on a 99.9% significance level for all currency denominations. In the last column of Table 2 one finds the estimated degrees of freedom for the Student t density for each of the currency denominations. One notes that the degrees of freedom are in the range from 2.2 to 4.8. We emphasize that the average estimated value 3.9899 of the degrees of freedom for log-returns of WSI currency denominations is very close to four, which is predicted by the minimal market model, see Section 5.

For the denomination of the WSI in units of different major currencies Table 2 confirms for the long period from 1970 until 2004 a Student t property of their log-returns, similar to that discovered in Markowitz & Usmen (1996a) for log-returns of the S&P500.

To visualize further the distributional nature of the WSI log-returns the estimated values of the shape parameters $\bar{\alpha}$ and λ from the SGH density are displayed in Figure 3 in an $(\bar{\alpha}, \lambda)$-scatter plot for the 34 WSI denominations in different

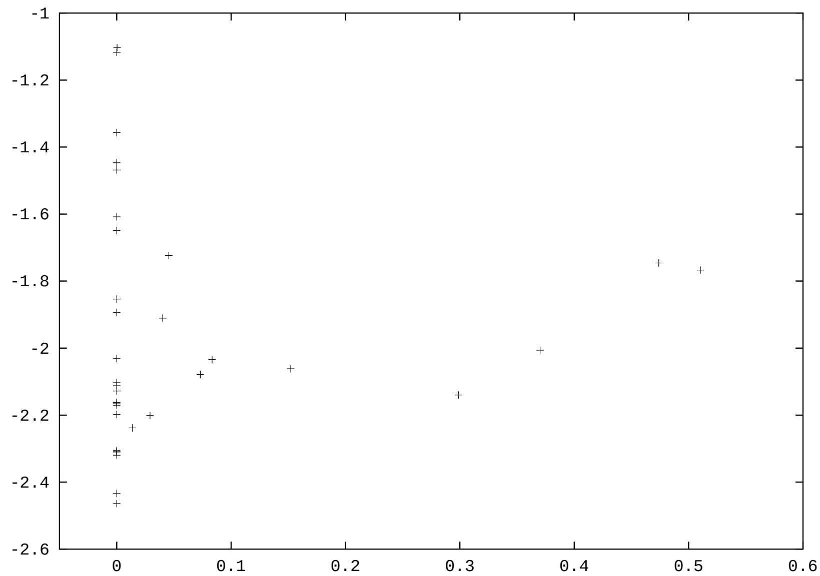

Figure 3: $(\bar{\alpha}, \lambda)$-plot for log-returns of WSI in different currencies.

currencies. Interestingly, the estimated $(\bar{\alpha}, \lambda)$ points are approximately localized near the negative lambda axis. It is the Student-t density that arises for $(\bar{\alpha}, \lambda)$-

parameter points near the negative λ-axis, whereas a variance-gamma density is obtained for $(\bar{\alpha}, \lambda)$-parameter points near the positive λ-axis. One notes that the cluster for the estimated parameter points is concentrated close to the point with coordinates $\bar{\alpha} = 0$ and $\lambda = -2$. This parameter choice refers to the Student t distribution with four degrees of freedom. For comparison, the normal-inverse Gaussian density refers in Figure 3 to the horizontal line with $\lambda = -0.5$. The hyperbolic density would give estimated parameters at the horizontal line $\lambda = 1.0$. Figure 3 visually confirms what already Table 2 revealed: The observed log-returns are likely to be Student t distributed and probably not normal-inverse Gaussian, hyperbolic or variance gamma distributed.

5 A Derivation of the Minimal Market Model

5.1 Continuous Benchmark Model

To explain our statistical findings we give in the following a derivation of the *minimal market model* (MMM) along the lines as described in Platen (2001, 2002, 2004a). For the modeling of a financial market we rely on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0, \infty)$, satisfying the usual conditions, see Karatzas & Shreve (1991). The trading uncertainty is expressed by the independent standard Wiener processes $W^k = \{W^k_t,\, t \in [0,T]\}$ for $k \in \{1, 2, \ldots, d\}$.

We consider a continuous financial market that comprises $d+1$ primary security accounts, $d \in \{1, 2, \ldots\}$. These include a savings account $S^{(0)} = \{S^{(0)}(t),\, t \in [0,T]\}$, which is a locally riskless primary security account whose value at time t is given by

$$S^{(0)}(t) = \exp\left\{\int_0^t r(s)\, ds\right\} \tag{5.1}$$

for $t \in [0,T]$, where $r = \{r(t),\, t \in [0,T]\}$ denotes the adapted short rate process. They also include d nonnegative, risky primary security account processes $S^{(j)} = \{S^{(j)}(t),\, t \in [0,T]\}$, $j \in \{1, 2, \ldots, d\}$, each of which contains units of one type of stock with all proceeds reinvested.

To specify the dynamics of primary securities in the given financial market we assume that the jth primary security account value $S^{(j)}(t)$, $j \in \{1, 2, \ldots, d\}$, satisfies the stochastic differential equation (SDE)

$$dS^{(j)}(t) = S^{(j)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} b^{j,k}(t) \left(\theta^k(t)\, dt + dW^k_t\right)\right) \tag{5.2}$$

for $t \in [0,T]$. Here $\theta^k(t)$ denotes the market price of risk with respect to the kth Wiener process. We assume that the volatility matrix $b(t) = [b^{j,k}(t)]^d_{j,k=1}$

is invertible for Lebesgue-almost every $t \in [0,T]$ with inverse matrix $b^{-1}(t) = [b^{-1\,j,k}(t)]_{j,k=1}^{d}$.

We call a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0,T]\}$ a *strategy* if for each $j \in \{0, 1, \ldots, d\}$ the Itô stochastic integral

$$\int_0^t \delta^{(j)}(s)\, dS^{(j)}(s) \tag{5.3}$$

exists, see Karatzas & Shreve (1991). Here $\delta^{(j)}(t)$, $j \in \{0, 1, \ldots, d\}$, is the number of units of the jth primary security account that are held at time $t \in [0,T]$ in the corresponding portfolio. We denote by

$$S^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, S^{(j)}(t) \tag{5.4}$$

the time t value of the portfolio process $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,T]\}$. A strategy δ and the corresponding portfolio $S^{(\delta)}$ are said to be *self-financing* if

$$dS^{(\delta)}(t) = \sum_{j=0}^{d} \delta^{(j)}(t)\, dS^{(j)}(t) \tag{5.5}$$

for $t \in [0,T]$. In what follows we consider only self-financing strategies and portfolios and will therefore omit the phrase “self-financing”.

Let $S^{(\delta)}$ be a portfolio process whose value $S^{(\delta)}(t)$ at time $t \in [0,T]$ is nonzero. In this case it is convenient to introduce the jth *fraction* $\pi_\delta^{(j)}(t)$ of $S^{(\delta)}(t)$ that is invested in the jth primary security account $S^{(j)}(t)$, $j \in \{0, 1, \ldots, d\}$, at time t. This fraction is given by the expression

$$\pi_\delta^{(j)}(t) = \delta^{(j)}(t)\, \frac{S^{(j)}(t)}{S^{(\delta)}(t)} \tag{5.6}$$

for $j \in \{0, 1, \ldots, d\}$. Note that fractions can be negative and always sum to one, that is

$$\sum_{j=0}^{d} \pi_\delta^{(j)}(t) = 1 \tag{5.7}$$

for $t \in [0,T]$. By (5.5), (5.2) and (5.6) we get for a nonzero portfolio value $S^{(\delta)}(t)$ the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t) \left(\theta^k(t)\, dt + dW_t^k\right) \right). \tag{5.8}$$

For a strictly positive portfolio $S^{(\delta)}$ we obtain for $\ln(S^{(\delta)}(t))$ the SDE

$$d\ln(S^{(\delta)}(t)) = g_\delta(t)\, dt + \sum_{k=1}^{d} b_\delta^k(t)\, dW_t^k \tag{5.9}$$

with *portfolio growth rate*

$$g_\delta(t) = r(t) + \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t)\, \theta^k(t) - \frac{1}{2} \left(\sum_{j=1}^{d} \pi_\delta^{(j)}(t)\, b^{j,k}(t) \right)^2 \right) \tag{5.10}$$

for $t \in [0,T]$. A strictly positive portfolio process $S^{(\delta_*)} = \{S^{(\delta_*)}(t),\, t \in [0,T]\}$ is called a *growth optimal portfolio* (GOP) if, for all $t \in [0,T]$ and all strictly positive portfolios $S^{(\delta)}$, the inequality

$$g_{\delta_*}(t) \geq g_\delta(t) \tag{5.11}$$

holds almost surely.

By using the corresponding first order conditions one can determine the *optimal fractions*

$$\pi_{\delta_*}^{(j)}(t) = \sum_{k=1}^{d} \theta^k(t)\, b^{-1\,j,k}(t) \tag{5.12}$$

for all $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$, which maximize the portfolio growth rate (5.10). It is straightforward to show in the given continuous financial market, see Long (1990), Karatzas & Shreve (1998) or Platen (2002), that a GOP value $S^{(\delta_*)}(t)$ satisfies the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t) \left(r(t)\, dt + \sum_{k=1}^{d} \theta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \right) \tag{5.13}$$

for $t \in [0,T]$ with $S^{(\delta_*)}(0) > 0$. From now on we use a GOP as *benchmark* and refer to the above financial market model as *continuous benchmark model.*

5.2 Optimal Portfolios

Given a strictly positive portfolio $S^{(\delta)}$, its *discounted value*

$$\bar{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(0)}(t)} \tag{5.14}$$

satisfies by (5.1), (5.8) and an application of the Itô formula the SDE

$$d\bar{S}^{(\delta)}(t) = \sum_{k=1}^{d} \psi_\delta^k(t) \left(\theta^k(t)\, dt + dW_t^k \right) \tag{5.15}$$

with

$$\psi_\delta^k(t) = \sum_{j=1}^{d} \delta^{(j)}(t)\, \frac{S^{(j)}(t)}{S^{(0)}(t)}\, b^{j,k}(t) \tag{5.16}$$

for $k \in \{1, 2, \ldots, d\}$ and $t \in [0, T]$. By (5.15) and (5.16) $\bar{S}^{(\delta)}$ has *discounted drift*

$$\alpha_\delta(t) = \sum_{k=1}^{d} \psi_\delta^k(t)\, \theta^k(t) \tag{5.17}$$

at time $t \in [0, T]$. The trading uncertainty of $\bar{S}^{(\delta)}$ at time $t \in [0, T]$ can be measured by its *aggregate diffusion coefficient*

$$\gamma_\delta(t) = \sqrt{\sum_{k=1}^{d} \left(\psi_\delta^k(t)\right)^2}. \tag{5.18}$$

We call a strictly positive portfolio $S^{(\tilde{\delta})}$ *optimal*, if for all $t \in [0, T]$ and all strictly positive portfolios $S^{(\delta)}$ when

$$\gamma_{\tilde{\delta}}(t) = \gamma_\delta(t) \tag{5.19}$$

we have

$$\alpha_{\tilde{\delta}}(t) \geq \alpha_\delta(t). \tag{5.20}$$

An optimal portfolio provides a characterization of the fact that *investors* prefer always *more to less*.

Let us introduce the *total market price of risk*

$$|\theta(t)| = \sqrt{\sum_{k=1}^{d} (\theta^k(t))^2}, \tag{5.21}$$

where we assume $|\theta(t)| \in (0, \infty)$ almost surely for all $t \in [0, T]$. In Platen (2002) it has been shown that the value $\bar{S}^{(\delta)}(t)$ at time t of a discounted, optimal portfolio satisfies the SDE

$$d\bar{S}^{(\delta)}(t) = \bar{S}^{(\delta)}(t) \frac{b_\delta(t)}{|\theta(t)|} \sum_{k=1}^{d} \theta^k(t) \left(\theta^k(t)\, dt + dW_t^k\right), \tag{5.22}$$

with optimal fractions

$$\pi_\delta^{(j)}(t) = \frac{b_\delta(t)}{|\theta(t)|}\, \pi_{\delta_*}^{(j)}(t) \tag{5.23}$$

for all $j \in \{1, 2, \ldots, d\}$ and $t \in [0, T]$. By (5.23) any optimal portfolio value can be decomposed into a fraction of wealth that is invested in the GOP and a remaining fraction that is held in the savings account.

We assume the existence of $n \in \{1, 2, \ldots\}$ investors who hold all investable wealth in the market and form each an optimal portfolio with their wealth. The optimal portfolio of investable wealth of the ℓth investor is denoted by $S^{(\delta_\ell)}$,

$\ell \in \{1, 2, \ldots, n\}$. The portfolio $S^{(\delta_{\mathrm{MP}})}(t)$ of the total investable wealth of all investors is then the *market portfolio* (MP)

$$S^{(\delta_{\mathrm{MP}})}(t) = \sum_{\ell=1}^{n} S^{(\delta_\ell)}(t) \tag{5.24}$$

at time $t \in [0, T]$. To identify the SDE of the MP we assume for all $t \in [0, T]$ that $S^{(\delta_{\mathrm{MP}})}(t) > 0$ almost surely and $\pi^{(0)}_{\delta_*}(t) \neq 1$.

The discounted MP $\bar{S}^{(\delta_{\mathrm{MP}})}(t)$ at time t is then determined by the SDE

$$\begin{aligned} d\bar{S}^{(\delta_{\mathrm{MP}})}(t) &= \sum_{\ell=1}^{n} \frac{\left(\bar{S}^{(\delta_\ell)}(t) - \delta^{(0)}_\ell(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\,(\theta^k(t)\,dt + dW^k_t) \\ &= \bar{S}^{(\delta_{\mathrm{MP}})}(t)\, \frac{\left(1 - \pi^{(0)}_{\delta_{\mathrm{MP}}}(t)\right)}{\left(1 - \pi^{(0)}_{\delta_*}(t)\right)} \sum_{k=1}^{d} \theta^k(t)\,(\theta^k(t)\,dt + dW^k_t) \end{aligned} \tag{5.25}$$

for $t \in [0, T]$. Consequently, the MP $S^{(\delta_{\mathrm{MP}})}(t)$ has the SDE of an optimal portfolio, which always holds a fraction of the wealth in the GOP and the remaining wealth in the savings account. If one assumes that at least in two different currency denominations the MP is an optimal portfolio, then it is straightforward to show that the MP must equal the GOP.

5.3 Minimal Market Model

We consider now the world stock index (WSI) $S^{(\delta_{\mathrm{WSI}})}$ of all investable stocks in the market. We have for the WSI zero holdings in the savings account, that is,

$$\pi^{(0)}_{\delta_{\mathrm{WSI}}}(t) = 0 \tag{5.26}$$

almost surely, for all $t \in [0, T]$. The WSI can be formed by a combination of holdings in the MP and the savings account. Therefore, the WSI satisfies the SDE of an optimal portfolio, which has by (5.22) and (5.25) the form

$$d\bar{S}^{(\delta_{\mathrm{WSI}})}(t) = \bar{S}^{(\delta_{\mathrm{WSI}})}(t)\, \frac{1}{1 - \pi^{(0)}_{\delta_*}(t)} \sum_{k=1}^{d} \theta^k(t)\left(\theta^k(t)\,dt + dW^k_t\right) \tag{5.27}$$

for all $t \in [0, T]$.

Now, we introduce the *discounted WSI drift*

$$\alpha_{\delta_{\mathrm{WSI}}}(t) = \bar{S}^{(\delta_{\mathrm{WSI}})}(t)\, \frac{|\theta(t)|^2}{1 - \pi^{(0)}_{\delta_*}(t)}, \tag{5.28}$$

see (5.17), as the average change per unit of time of the *underlying value* of the discounted WSI. We get then the total market price of risk in the form

$$|\theta(t)| = \sqrt{\frac{\alpha_{\delta_{\rm WSI}}(t)\,(1-\pi^{(0)}_{\delta_*}(t))}{\bar{S}^{(\delta_{\rm WSI})}(t)}}. \tag{5.29}$$

This yields by (5.27) and (5.29) for the discounted WSI the SDE

$$d\bar{S}^{(\delta_{\rm WSI})}(t) = \alpha_{\delta_{\rm WSI}}(t)\,dt + \sqrt{\frac{\bar{S}^{(\delta_{\rm WSI})}(t)\,\alpha_{\delta_{\rm WSI}}(t)}{(1-\pi^{(0)}_{\delta_*}(t))}}\,dW_t, \tag{5.30}$$

where

$$dW_t = \frac{1}{|\theta(t)|}\sum_{k=1}^{d}\theta^k(t)\,dW_t^k \tag{5.31}$$

is the stochastic differential of a standard Wiener process W for $t\in[0,T]$. The solution of the SDE (5.30) is a *time transformed squared Bessel process of dimension*

$$\nu(t) = 4\,(1-\pi^{(0)}_{\delta_*}(t)), \tag{5.32}$$

see Revuz & Yor (1999). Its *transformed time* $\varphi(t)$ equals the accumulated underlying value of the discounted WSI and is at time t given by the expression

$$\varphi(t) = \varphi(0) + \int_0^t \alpha_{\delta_{\rm WSI}}(s)\,ds \tag{5.33}$$

with $\varphi(0)\geq 0$.

Let us now introduce the *normalized WSI*

$$Y(t) = \frac{\bar{S}^{(\delta_{\rm WSI})}(t)}{\alpha_{\delta_{\rm WSI}}(t)}, \tag{5.34}$$

which satisfies by the Itô formula the SDE

$$dY(t) = dt + Y(t)\,\alpha_{\delta_{\rm WSI}}(t)\,d\left(\frac{1}{\alpha_{\delta_{\rm WSI}}(t)}\right) + \sqrt{\frac{Y(t)}{(1-\pi^{(0)}_{\delta_*}(t))}}\,dW_t + d\left\langle \bar{S}^{(\delta_{\rm WSI})}, \frac{1}{\alpha_{\delta_{\rm WSI}}}\right\rangle_t \tag{5.35}$$

for $t\in[0,T]$. We now make the reasonable assumption that we have a *net growth rate*

$$\eta_t = -\alpha_{\delta_{\rm WSI}}(t)\,\frac{d}{dt}\left(\frac{1}{\alpha_{\delta_{\rm WSI}}(t)}\right)$$

and zero covariation

$$\left\langle \bar{S}^{(\delta_{\rm WSI})}, \frac{1}{\alpha_{\delta_{\rm WSI}}}\right\rangle_t = 0$$

for all $t\in[0,T]$. Then $Y=\{Y(t),\, t\in[0,T]\}$ is a square root process with SDE

$$dY(t) = (1-\eta_t\,Y(t))\,dt + \sqrt{\frac{Y(t)}{(1-\pi^{(0)}_{\delta_*}(t))}}\,dW_t \tag{5.36}$$

with dimension $\nu(t)$ as given in (5.32).

In the case when the net growth rate η_t and the fraction $\pi^{(0)}_{\delta_*}(t)$ are constant, the square root process Y is known to have as stationary density a gamma density with $\nu(t)$ degrees of freedom, see Karatzas & Shreve (1991). The squared volatility of the WSI equals then by (5.30) and (5.34) the expression

$$\frac{1}{Y(t)\,(1-\pi^{(0)}_{\delta_*}(t))},$$

which has as stationary distribution an inverse gamma distribution with $\nu(t) = 4(1-\pi^{(0)}_{\delta_*}(t))$ degrees of freedom. Consequently, the mixing distribution for the variance of the log-returns of the WSI is an inverse gamma distribution. For a sufficiently long period of WSI log-return observations one obtains then the Student t distribution as estimated normal-variance mixture distribution with $4(1-\pi^{(0)}_{\delta_*}(t))$ degrees of freedom. This is what we have also found in our empirical study.

Moreover, we have estimated on average the degrees of freedom $3.9899 \approx 4.0$. For the case $\nu(t) = 4.0$ it follows from (5.32) that $\pi^{(0)}_{\delta_*}(t) = 0$. This then tells us that the WSI equals the GOP in this case. Therefore, the holdings in the savings account for the market portfolio amount in total to zero. This is consistent with the fact that for short term bonds there is always a buyer and a seller among the market participants.

Conclusion

The log-return distribution of a world stock index denominated in different currencies has been identified in the class of symmetric generalized hyperbolic distributions as a Student t distribution with about four degrees of freedom. This empirical stylized fact can be naturally explained by a version of the minimal market model.

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