Project `normal_AR3` notes

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These are some technical notes on the nonlinear mapping for the SABL toolbox project `normal_AR3`.

The starting point is my 1988 JBES paper. The AR3 model characteristic function $1 - \beta_1 z - \beta_2 z^2 - \beta_3 z^3$ has roots r_1, r_2, r_3 where r_1 is real and (r_2, r_3) is a complex conjugate pair. Using notation and results from the paper, $\alpha_s = |r_1|^{-1}$, $\alpha_c = |r_2|^{-1}$, $p = 2\pi / \tan^{-1}(\mathrm{Im}(r_2) / \mathrm{Re}(r_2))$, where $\tan^{-1}$ is the main $(0, \pi)$ branch of the inverse tangent function.

We will parameterize in terms of $\alpha_1 > 0$ (and take $r_1 > 0$), α_c and p. Moving to polar representation,

$$\begin{aligned} r_1 &= \alpha_s^{-1},\ r_2 = \alpha_c^{-1}\exp(i\pi\theta),\ r_3 = \alpha_c^{-1}\exp(-i\pi\theta); \\ \mathrm{Re}(r_2) &= \alpha_c^{-1}\cos(\theta),\ \mathrm{Im}(r_2) = \alpha_c^{-1}\sin(\theta),\ \mathrm{Im}(r_2)/\mathrm{Re}(r_2) = \tan(\theta). \end{aligned}$$

The period associated with θ is $p = 2\pi/\theta$, hence $\theta = 2\pi/p$. So (to re-derive part of the JBES paper as a check on the setup)

$$\mathrm{Im}(r_2)/\mathrm{Re}(r_2) = \tan(2\pi/p) \implies p = p = 2\pi/\tan^{-1}(\mathrm{Im}(r_2)/\mathrm{Re}(r_2)).$$

The polynomial $B(z) = 1 - \beta_1 z - \beta_2 z^2 - \beta_3 z^3$ has roots r_1, r_2, r_3 and leading term 1, so

$$\begin{aligned} B(z) &= (r_1 - z)(r_2 - z)(r_3 - z)/r_1 r_2 r_3 \\ &= \frac{(\alpha_s^{-1} - z)\left[\alpha_c^{-1}\exp(2i\pi/p) - z\right]\left[\alpha_c^{-1}\exp(-2i\pi/p) - z\right]}{\alpha_s^{-1}\alpha_c^{-1}\exp(2i\pi/p)\,\alpha_c^{-1}\exp(-2i\pi/p)} \\ &= \frac{(\alpha_s^{-1} - z)\left[\alpha_c^{-2} - 2\alpha_c^{-1}\cos(2\pi/p) + z^2\right]}{\alpha_s^{-1}\alpha_c^{-2}} \\ &= \frac{(\alpha_s^{-1} - z)\left[\alpha_c^{-2} - 2\alpha_c^{-1}\cos(2\pi/p)\right] z + \left[2\alpha_c^{-1}\cos(2\pi/p) + \alpha_s^{-1}\right] z^2 - z^3}{\alpha_s^{-1}\alpha_c^{-2}} \\ &= 1 - \left[\alpha_s + 2\alpha_c\cos(2\pi/p)\right] z + \left[2\alpha_s\alpha_c\cos(2\pi/p) + \alpha_c^2\right] z^2 - \alpha_s\alpha_c^2 z^3. \end{aligned}$$

Thus the final result used in the code

$$\begin{aligned} \beta_1 &= \alpha_s + 2\alpha_c\cos(2\pi/p), \\ \beta_2 &= -\left[2\alpha_s\alpha_c\cos(2\pi/p) + \alpha_c^2\right], \\ \beta_3 &= \alpha_s\alpha_c^2. \end{aligned}$$